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Numerical Modeling of the Coupled Hydro-Mechanical Response of Partially Saturated Soils under Cyclic and Dynamic Loading

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To my family....

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This dissertation is submitted in partial fulfilment of the requirements for the award of the Doctor of Philosophy degree with the additional label of International Doctor.

Declaration

I hereby declare that except where specific reference is made to the work of others, the contents of this dissertation are original and have not been submitted in whole or in part for consideration for any other degree or qualification in this, or any other university. This dissertation is my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified in the text and Acknowledgements.

Luis Rodrigo Perez Pinto

May 2026

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Abstract

This thesis develops and implements a computational framework for the coupled hydro-mechanical analysis of saturated and partially saturated soils within the OpenSees platform. The work addresses the need for robust numerical tools capable of simulating cyclic loading conditions in partially saturated porous media, where suction-dependent hydraulic and mechanical interactions significantly influence soil response.

An unsaturated constitutive formulation suitable for cyclic loading is first implemented and verified in an independent MATLAB-based constitutive driver. Based on the observed numerical performance and stability under cyclic conditions, a simplified and computationally robust variant is subsequently derived within a stress-ratio plasticity framework inspired by SANISAND/Manzari-Dafalias concepts with isotropic hardening. This formulation, incorporating suction-dependent effects governed by a van Genuchten retention law, is then implemented in OpenSees under the name `ManzariDafaliasIsoHard`.

In parallel, the governing equations for partially saturated porous media are formulated in a mixed finite element framework. An extended four-node quadrilateral element (`FourNodeQuadUPSVG`) is developed from the classical `QuadUP` ($u-p_w$) formulation to incorporate suction-dependent saturation and permeability while preserving numerical stability and compatibility with existing OpenSees solution strategies.

The proposed framework is verified through a comprehensive set of static, transient, and dynamic boundary value problems, including one-dimensional consolidation, partially saturated column analyses under transient hydraulic conditions, seepage through confined domains and earth-dam configurations, and dynamic site response under base excitation. The results demonstrate numerical robustness, stable integration under cyclic loading, and consistency with reference solutions, supporting the applicability of the developed constitutive-element formulation for advanced coupled analyses in geotechnical engineering where partial saturation plays a critical role.

Keywords: coupled hydro-mechanical modelling; finite element method; unsaturated soil mechanics; cyclic loading; dynamic analysis.

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Chapter 1

Introduction

1.1 Background of the Research

The mechanical behaviour of soils under static, quasi-static, and dynamic loading conditions plays a fundamental role in geotechnical engineering applications. In particular, unsaturated soils, which are commonly encountered in near-surface layers, earth structures, and foundation systems, exhibit a complex hydro-mechanical response governed by the interaction between solid, liquid, and gaseous phases [51]. This multi-phase interaction directly influences stress transfer mechanisms, deformation behaviour, and fluid flow processes in porous media, making hydro-mechanical coupling a central aspect of geotechnical analysis [27].

A realistic representation of the static and quasi-static hydro-mechanical state of unsaturated soils is essential, as it defines the initial stress conditions, degree of saturation, and pore pressure distributions that control subsequent cyclic and dynamic responses. Consequently, advanced constitutive models formulations are required to ensure consistency across different loading regimes. In this context, the finite element formulation plays a central role, as it provides the numerical framework through which coupled field variables are discretized, interpolated, and solved at the boundary value problem level.

Site response analysis represents one of the most common problems to deal with in geotechnical earthquake engineering (see, for instance, [80, 32, 43, 44, 38, 35]). The latter is adopted, for example, to predict surface ground motions in order to (i) develop design response spectra; (ii) determine earthquake-induced forces that can produce instability of retaining structures; (iii) evaluate the dynamic stresses and strains induced in soils [79] and in structures interacting with soil. With reference to this aspect, reliable constitutive models are required to reproduce the hysteretic response of soils subjected to dynamic excitation. Various frameworks are available for this purpose. Among them, multisurface plasticity [124], [159] and bounding surface plasticity [30, 117] revealed particularly effective. Models based

on the above mentioned theories are formulated in terms of effective stresses and reliably describe complex problems, such as the nonlinear dynamic response of saturated soils [126]. In fact, although soil is physically composed of three phases (i.e., solid, liquid, and gaseous; [48]), classical soil mechanics has traditionally idealized it as a two-phase system (solid and liquid), which may be adequate under fully saturated conditions but becomes restrictive when the soil is partially saturated.

In reality, most near-surface soils encountered in geotechnical engineering are in a partially saturated state, where both water and air coexist within the pore space. Under these conditions, matric suction influences stiffness, strength, and volumetric response. For this reason, unsaturated soil mechanics introduces the air phase explicitly in order to provide a more realistic representation of soil behaviour in geotechnical simulations.

From a computational standpoint, the objective is not to increase model complexity unnecessarily, but to incorporate the essential hydro-mechanical coupling mechanisms within a consistent numerical framework. As shown in previous implementations developed for saturated soils (see, for instance, [148, 5, 42, 41, 17, 25]), the integration of constitutive models into a computational code allows the governing equations to be solved in a stable and efficient manner.

In this context, the development of appropriate finite elements is fundamental, since the element formulation governs the spatial discretization of displacement and pore-pressure fields, the treatment of coupling terms, and the overall numerical robustness of the solution, independently of the specific constitutive law adopted. Furthermore, the behaviour of unsaturated soils is related to various geotechnical engineering applications, including slope stability, shallow and deep foundations, pavements, earth structures, retaining walls, buried structures, and geoenvironmental applications [125], so there is significant interest in understanding this behaviour, even considering its response under cyclic loads. In fact, superficial layers of soil interacting with many earth structures, such as pavements and railway embankments, are generally unsaturated and experience cyclic loads; consequently, the adoption of unsaturated soil mechanics is necessary to adequately calculate settlements and guarantee the serviceability limit state design [97, 115, 169]. The aim is to widely implement this knowledge in practice [66]. This broad range of applications highlights the need for numerical tools capable of addressing coupled hydro-mechanical processes under diverse loading conditions within a unified framework. As a matter of fact, for nearly 60 years, there has been rapid growth in the field of unsaturated soil mechanics, including the advancement of unsaturated soil theories, the measurement of unsaturated soil properties, and the development of constitutive models [49]. In earthquake engineering, unsaturated soil mechanics has been studied since the 1970s. For instance, some pioneering works can

be found in Sherif et al.[132] or Chaney [24], in which they analysed the influence of the degree of saturation on liquefaction resistance and shear strength, respectively. Therefore, there is interest in predicting the correct dynamic behaviour of unsaturated soils since many geotechnical earthquake problems can be addressed (e.g., soil-structure interaction, liquefaction, and earthquake resistance of structures; [165]). For this purpose, many advances have been developing: for example, in laboratory testing, Altun and Goktepe [2] studied the stress-strain behaviour and strain softening response of partially saturated cohesive soil under undrained conditions in a cyclic torsional shear apparatus; Khosravi and McCartney [75] performed resonant column tests for unsaturated soils with suction and saturation control; Eseller-Bayat et al. [36] investigated the liquefaction response of partially saturated loose sands, including the design of a cyclic simple shear liquefaction box equipped with pore pressure and displacement transducers, as well as bender elements and bending disks; Khosravi et al. [76] studied the effect of hydraulic hysteresis on the small-strain shear modulus of unsaturated sand. Conversely, considering in-situ conditions, Yang [156] investigated the effects of saturation state on Rayleigh's wave velocity in idealized partially saturated sand, while Whalley et al. [151] measured the shear wave velocity in sands at different values of matric suction and confining pressures. Based on observed behaviour in laboratory and field conditions, constitutive models have been developed to simulate the stress-strain response of unsaturated soils under cyclic conditions for numerical analysis and only a few of them were implemented in finite element (FE) or finite difference (FD) codes [91, 154]. In fact, implementation within software frameworks is required to evaluate constitutive models' performance in site response analysis [39, 40]. The inclusion of constitutive models should be performed together with the implementation of fully coupled finite elements that takes into account for the dynamic behaviour of unsaturated soils. Accordingly, the finite element itself constitutes a key component of the numerical model, as it enables the consistent integration of constitutive behaviour, governing equations, and boundary conditions within a unified computational framework [165].

1.2 Position of the problem

From a numerical modelling standpoint, the analysis of soil behaviour under different loading conditions represents a relevant challenge in geotechnical engineering. In this context, local soil conditions exercise a strong impact on soil response, and these effects are commonly evaluated through boundary value problem analyses associated with a wide range of geotechnical applications. Among these, problems involving cyclic and dynamic loading constitute a particularly relevant class, and earthquake-induced ground response represents

one important example that is commonly investigated through site response analysis. In this framework, it has been shown that the degree of saturation, among other parameters such as strain level, loading pattern, and confining pressure, influences the dynamic properties of the soil (i.e., shear modulus and damping [106, 107, 105]). However, Mirshekari [105] stated that the introduction of partial saturation has not been incorporated in site response analysis in practice yet, although in the literature there are many works that discussed the effect of matric suction in this class of problems (e.g., [158, 90, 157, 145, 146, 34, 114, 14, 13, 59]).

On the contrary, despite evidence from experimental studies suggesting that partial saturation can significantly influence the dynamic response of soils, for example by modifying stiffness and damping characteristics and, in some cases, being associated with greater amplification of ground motion and enhanced resistance to liquefaction compared with fully saturated conditions [2], site response analyses have commonly been performed using the simplifying assumption of soil in either dry or fully saturated states [106, 59]. Site response analysis can be performed by considering linear, equivalent-linear, and nonlinear soil behaviour in one-, two-, and three-dimensional settings [107, 12]. However, Mirshekari and Ghayoomi [106] and Ravichandran et al. [128] highlighted that, despite the development of numerous numerical finite element software tools for conducting site response analysis (e.g., SHAKE; DESRA; DMOD; DEEPSOIL; PLAXIS; and OpenSees), the direct assessment of the influence of partial saturation in site response analysis using commercial or research-oriented software is still not implemented.

To address this gap, researchers have developed and implemented constitutive models together with finite element formulations accounting for partial saturation [128]. For instance, in the analysis features of the Real-ESSI Simulator software [68] and U-DYSAC2 v2 [110], the option to perform a partially saturated site response analysis is available. Nevertheless, the implementation of such approaches within general-purpose finite element frameworks, in which the finite element formulation governs the discretization of coupled field variables and their numerical interaction, together with their systematic verification through boundary value problems of increasing complexity, remains limited.

1.3 Objective

The primary aim of this research is to formulate, implement, and verify a coupled hydro-mechanical numerical framework for partially saturated soils under cyclic and dynamic loading, including the implementation of an advanced constitutive model and its integration within a finite element formulation, with application to geotechnical boundary value problems.

- Implement the constitutive model developed by Liu and Muraleetharan [87] using an explicit integration scheme within a Constitutive Driver environment, in order to assess its hydro-mechanical response at the material point level.
- Implement and extend a U–P finite element formulation within the OpenSees framework to incorporate partial saturation effects through the introduction of pore air pressure, enabling a consistent three-phase hydro-mechanical description at the element level.
- Integrate the implemented constitutive model within the developed finite element formulation, ensuring consistency between the constitutive response, the interpolation of field variables, the governing equations, and the numerical solution strategy.
- Assess the numerical performance and robustness of the proposed formulation through element-level and material point simulations, including point response analyses using the Constitutive Driver and element-level tests within the finite element framework.
- Verify the capability of the coupled finite element formulation to reproduce hydro-mechanical processes in boundary value problems under static and hydraulic loading conditions, including one-dimensional soil column analyses and seepage problems.
- Extend the verification to boundary value problems of increasing complexity, including earth dam simulations under coupled hydro-mechanical conditions.
- Evaluate the performance of the proposed numerical framework under cyclic and dynamic loading conditions, including simulations representative of earthquake-induced loading.

Chapter 2

Cyclic Behavior and Constitutive Modeling of Unsaturated Soils

2.1 Fundamentals of Unsaturated Soil Mechanics

Soil is generally composed of solid particles (skeleton), with voids containing both liquid and gaseous phases. In geotechnical engineering, these phases are typically assumed to be water and air, respectively. By neglecting the gaseous phase, Classical Soil Mechanics emerged, providing fundamental insights into soil behavior and contributing to the development of solutions for geotechnical engineering challenges over the past century [48]. However, as soil mechanics advanced, the necessity of accounting for the third phase became increasingly evident. Rather than being a specialized branch of Classical Soil Mechanics, Unsaturated Soil Mechanics establishes a comprehensive framework that integrates all three phases, offering a more general and complete approach to understanding soil behavior.

2.1.1 The Unsaturated Zone

In Geotechnical Engineering, a soil profile consists of a zone beneath the water table (i.e., the Saturated Zone) and a zone above it (i.e., the Vadose Zone), where the soil is partially saturated and the pore water pressure is negative (i.e., less than atmospheric pressure) [113]. According to Fredlund [48], while the Saturated Zone consists of two phases (i.e., soil particles and water), the Vadose Zone comprises four phases (i.e., soil particles, air, water, and the contractile skin) (see Figure (2.1)). It is important to note that both fluid phases can exist within the voids either as a continuous phase or as isolated portions surrounded by the other fluid. In the case of air, it can be categorized into air and occluded air, whereas water can be classified as bulk water and meniscus water [152].

Many geotechnical engineering challenges arise within the Vadose Zone, and its depth can vary significantly in arid regions, primarily depending on the thermodynamic interactions between surrounding perturbations on the ground surface (e.g., local or regional climate) and the water table [113]. For example, during dry periods, the depth of the Vadose Zone can extend to hundreds of meters, whereas during wet periods, it may become significantly shallower [48]. However, understanding the Vadose Zone is not only relevant in arid regions, as climate change is expected to alter environmental conditions globally, exposing many regions to unprecedented climate changes[141]. Furthermore, soil used as a construction material in many civil engineering works is very often placed in partially saturated conditions and remains in this state throughout the service life of the structure.

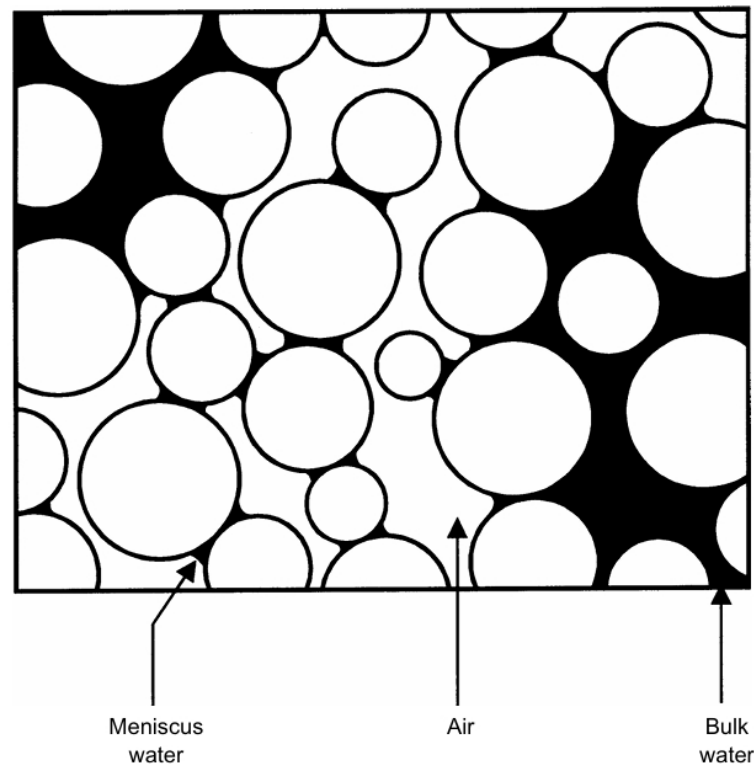


Fig. 2.1 Soil phases extracted from Wheeler et al. [152].

2.1.2 Total Suction

According to Lu and Likus [113], the energy required to extract a water molecule from the pores and convert it into free water is referred to as total suction. Total suction is influenced by generally capillary forces and, in some cases, by the presence of solutes in the water. Thus,

it can be expressed as the sum of matric suction and osmotic suction, respectively (Equation 2.1).

$$\psi = (u_a - u_w) + \pi \quad (2.1)$$

In this work, the term total suction will be considered equivalent to matric suction ($u_a - u_w$), given that the influence of osmotic suction (π) in constitutive modeling has not been widely accounted for in most cases.

2.1.3 Stress State Variables

In this work, the term total suction is considered equivalent to matric suction ($u_a - u_w$), given that the influence of osmotic suction (π) in constitutive modeling has not been widely incorporated in most practical formulations.

The effective stress principle proposed by Terzaghi [139] represented a major advance in geotechnical engineering, as it established a clear mechanical framework for saturated soils. This principle is based on two state variables: the total stress tensor ($\boldsymbol{\sigma}$) and the pore-water pressure (u_w).

For partially saturated soils, several extensions have been proposed. Among them, the generalized effective stress expression introduced by Bishop [15] incorporates the pore-air pressure and a weighting parameter χ to account for partial saturation effects. From a thermodynamic perspective, independent stress-variable frameworks such as those proposed by Houlsby [65] and further discussed more recently formalized by Buscarnera [18], provide a rigorous basis for defining work-conjugate stress variables in unsaturated media and for ensuring mathematical and thermodynamic consistency of elastoplastic models.

Within this context, Jommi [70] demonstrated that the average soil skeleton stress may be consistently expressed by adopting Bishop's formulation with the parameter χ taken equal to the degree of saturation ($\chi = S_r$). This choice provides a physically meaningful interpolation between dry and saturated states and represents a particular, yet widely adopted, case compatible with thermodynamic consistency under appropriate assumptions.

Accordingly, in the present work Bishop's effective stress is adopted in the form:

$$\boldsymbol{\sigma}' = \boldsymbol{\sigma} - u_a \mathbf{I} + \chi (u_a - u_w) \mathbf{I} \quad (2.2)$$

where $\boldsymbol{\sigma}'$ is the effective stress tensor; $\boldsymbol{\sigma}$ is the total stress tensor; u_a is the pore air pressure; u_w is the pore water pressure; χ is Bishop's parameter, which varies with the degree of saturation (i.e., $\chi = 0$ for dry conditions and $\chi = 1$ for fully saturated conditions); and $\mathbf{I}$ is the second-order identity tensor.

Bishop's approach has garnered numerous unfavourable assessments, which have been deliberated in many works, including Gens et al. [56] and Fredlund [47]. Although Zhang and Lytton [166] and Leong [83] indicate that it is still unclear how many stress state variables should be considered to study partially saturated soils, Gens et al. [56] and Fredlund [47] concluded that for a better understanding of the behaviour of unsaturated soil, at least two constitutive variables must be considered (e.g., net normal stress $((\sigma - u_a))$ and the matric suction $(u_a - u_w)$). The ambiguity in the selection of the type of the constitutive variables is reflected in the formulation of the constitutive models for partially saturated response. For this reason, Gens [57] proposed grouping them into a general framework, categorizing them as First Constitutive Variables (FCV) and Second Constitutive Variables (SCV) (**FCV** and **SCV**, respectively) as expressed by Equation (2.3).

$$\text{FCV: } \sigma_{ij} - u_a \delta_{ij} + u_1(s, S_r) \delta_{ij} \quad \text{SCV: } u_2(s, S_r) \delta_{ij} \quad (2.3)$$

The **FCV** is responsible for explaining the soil skeleton behaviour under changes in loading and is represented as a tensor. On the other hand, the **SCV** is proposed to capture the effect of suction in both the bulk water and meniscus water and is usually represented as either a scalar or a tensor.

2.1.4 Soil Water Retention Curve

The Soil Water Retention Curve (S.W.R.C) is an important constitutive law in unsaturated soil mechanics, particularly within constitutive models. It enables to establish a connection between changes in volume or stress history and variations in suction [72, 104]. Even more importantly, it helps establishing a link between mechanical and hydraulic properties. The SWRC is usually represented in terms of gravimetric content, volumetric content, or degree of saturation versus matric suction as it is presented in Figure (2.2). Generally, the SWRC consists of three distinct zones: the boundary effect zone, the transition zone, and the residual zone of unsaturation. The boundaries between these zones are defined by the air-entry value and the residual water content, respectively.

It is worth noting that as the soil undergoes drying and wetting processes, hydraulic hysteresis can occur. Due to hysteresis, there are three primary SWRC bounding curves designated as initial drying curve, main drying, and wetting curves (see Figure (2.3)) [167, 52, 50]. Between these bounding curves, infinite number of SWRC scanning curves can be generated due to cyclic processes. The differences among the bounding curves are due to the ink-bottle effect, the contact angles in the meniscus, entrapped air, and thixotropic regain [121].

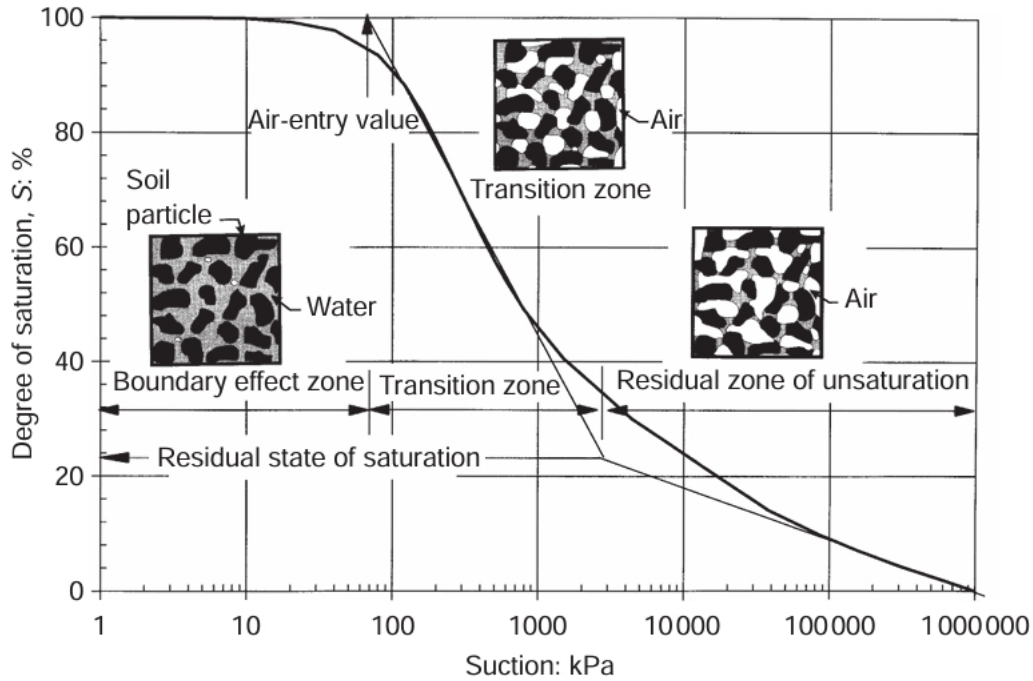


Fig. 2.2 Hysteresis extracted from Vanapalli et al. [144].

Since the initial mathematical formulations to represent the SWRC curves (i.e., boundary curves and scanning curves), numerous features have been progressively incorporated into them over time. For instance, Van Genuchten [143] and Fredlund and Xing [53] proposed to add the pore size distribution of the soil in the expression of the curve. Mašín [94], Khoshghalb and Khalili [73], Khoshghalb et al. [74] and Pasha et al. [118] integrated the void ratio influence.

The van Genuchten model [143] is described here, as it is used in this work to relate the degree of saturation S_r to the matric suction $s = u_a - u_w$ as:

$$S_r(s) = S_r^{\text{res}} + (1 - S_r^{\text{res}}) \left[\frac{1}{(1 + (\alpha_{VG}s)^{n_{VG}})^{m_{VG}}} \right] \quad (2.4)$$

$$m_{VG} = 1 - 1/n_{VG} \quad (2.5)$$

where S_r^{res} is the residual degree of saturation, α_{VG} is the inverse of the air-entry value, and n_{VG} and m_{VG} are empirical fitting parameters.

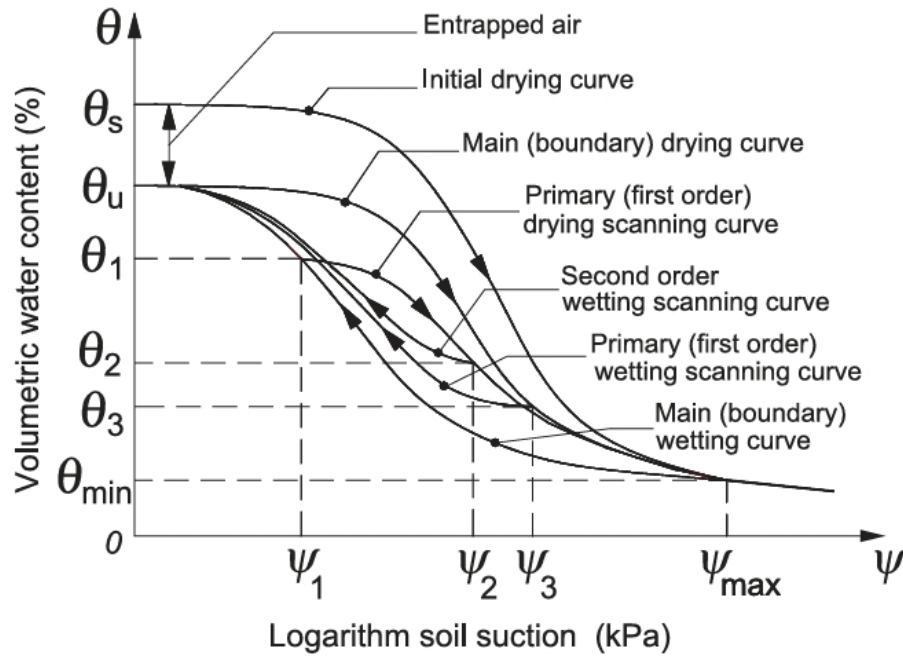


Fig. 2.3 SWRC extracted from Pham et al. [121].

2.2 Cyclic Behavior

In contrast to the widespread recognition of the importance of considering partially saturated soils in earthquake engineering practice, research into their cyclic and dynamic behaviour remains relatively unexplored. As it is stated in Zhou [167], previous studies on cyclic behaviour of partially saturated soils focused on three main themes: resilient modulus, plastic strain accumulation and liquefaction potential. The resilient modulus, particularly important in pavement design, was defined by Seed et al. [131] as the ratio between the cyclic deviator stress – constant for cyclic conditions – and the axial recoverable strain. The higher is the axial recoverable strain, the lower is the resilient modulus. Less attention is given to the plastic strain accumulation, likely due to the time-consuming process required to monitor plastic strain accumulation over a large number of cycles [167]. Finally, liquefaction potential of unsaturated soils has recently attracted the interest of a large number of researchers.

2.2.1 Liquefaction of Unsaturated Soils

Earthquakes can induce several co-seismic ground effects, including landslides, liquefaction, and soil densification [46] [45]. Among them, soil liquefaction is one of the most complex and historically controversial co-seismic effects that can take place as a consequence of an earthquake [81]. The phenomenon is extensively documented in the scientific literature,

and various catastrophic consequences were observed during recent earthquakes ([127]; [21]; [163]; [10]), such as extreme ground settlements, building collapse, and disruptions to lifelines.

Liquefaction was initially believed to occur only in fully saturated soils. However, recent studies have demonstrated that even partially saturated soils can experience liquefaction under certain conditions. For instance, Okamura and Soga [116] investigated the liquefaction potential of unsaturated sand through undrained cyclic triaxial tests. Further experimental evidence on liquefaction triggering in unsaturated soils has been provided by Yoshimi et al. [161], Unno et al. [142], Arab et al. [6], Yegian et al. [160], and Mele et al. [103]. In particular, Unno et al. [142] adopted Bishop's effective stress framework to identify the onset of liquefaction [167], which is defined to occur when the effective stress approaches zero. Field evidence also supports this phenomenon, as observed during the Miyagi, Japan earthquake [8].

2.3 Constitutive Modeling of Unsaturated Soils

There is considerable interest in gaining insight into the impact of saturation level and consequent alterations in volumetric deformations on mechanical properties using hydro-mechanical constitutive models [58]. This interest arises not only from experimental evidence highlighting the need for coupling hydraulic and mechanical behaviour but also from the interactions among the different phases of unsaturated soil, such as air, water, and the solid skeleton [87]. Since the early development of constitutive models for partially saturated soils, significant advancements have been continuously incorporated to more accurately represent the real conditions to which soils are subjected, particularly regarding their mechanical and hydraulic properties.

The development of constitutive laws for partially saturated soils under monotonic loads is based on elasto-plasticity theory [56]. However, for simulating cyclic behaviour, additional features must be included, such as the incorporation of isotropic and kinematic hardening. The bounding surface approach has proven highly valuable for advancements in this regard ([92]) enabling the accurate representation of a hysteretic response, even considering the past stress history, gradual stiffness degradation during loading, preservation of small strain stiffness, and early onset of plastic strains ([55, 72]). Additionally, the bounding surface approach provides simplicity and efficiency in cyclic numerical modeling, even from an implementation standpoint.

Given its importance and wide applicability in modern constitutive formulations, the following section first revisits the fundamental aspects of plasticity theory and concludes with an explanation of the bounding surface theory.

2.3.1 Basic Aspects of Plasticity

A brief review of the plasticity theory is presented since it is needed later for the formulation of advanced constitutive models. It is worth noting that tensors are written in bold. The general plasticity theory is described in a wide number of works, such as Lubliner [89], Simo and Hughes [133] and Jirasek and Bazant [69], among others, while its application to geomechanics is reported in Vermeer and de Borst [147], Zienkiewicz et al. [177], Potts and Zdravkovic [164], Wood [153], and Yu [162]. Considering the aspects shown in the following, the interested reader can refer, among others, to Potts and Zdravkovic [164], Wood [153] and Yu [162] for further and more detailed information.

2.3.1.1 Tensors and Strains

Stresses and strain tensors adopted for the models are reported in the following equations. Stresses are effective by default and the prime symbol was removed. The superscripts e and p refer to elastic and plastic components of strain, respectively, while the general stress and strain tensors are expressed in the (x, y, z) space by σ_{ij} and ε_{ij} . The terms referred as σ_1 , σ_2 and σ_3 indicate principal stresses. The basic stress tensors are reported in 2.6.

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \quad (2.6)$$

With mean stress p indicated as 2.7:

$$p = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \quad (2.7)$$

Deviatoric stress tensor $\mathbf{s}$:

$$\mathbf{s} = \boldsymbol{\sigma} - p\mathbf{I} = \begin{bmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{yx} & s_{yy} & s_{yz} \\ s_{zx} & s_{zy} & s_{zz} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} - p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} - p & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} - p \end{bmatrix} \quad (2.8)$$

Deviatoric stress ratio tensor $\mathbf{r}$:

$$\mathbf{r} = \frac{\mathbf{s}}{p} = \begin{bmatrix} r_{xx} & r_{xy} & r_{xz} \\ r_{yx} & r_{yy} & r_{yz} \\ r_{zx} & r_{zy} & r_{zz} \end{bmatrix} = \begin{bmatrix} \frac{\sigma_{xx} - p}{p} & \frac{\sigma_{xy}}{p} & \frac{\sigma_{xz}}{p} \\ \frac{\sigma_{yx}}{p} & \frac{\sigma_{yy} - p}{p} & \frac{\sigma_{yz}}{p} \\ \frac{\sigma_{zx}}{p} & \frac{\sigma_{zy}}{p} & \frac{\sigma_{zz} - p}{p} \end{bmatrix} \quad (2.9)$$

Where $\boldsymbol{\sigma}$, $\mathbf{s}$, and $\mathbf{r}$ are symmetric tensors. The basic strain tensors are defined in 2.6–2.8:
Strain tensor $\boldsymbol{\varepsilon}$:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \quad (2.10)$$

With volumetric strain referred as 2.11:

$$\varepsilon_v = \text{tr}(\boldsymbol{\varepsilon}) = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \quad (2.11)$$

Deviatoric strain tensor $\mathbf{e}$:

$$\mathbf{e} = \boldsymbol{\varepsilon} - \frac{\varepsilon_v}{3} \mathbf{I} = \begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} - \frac{\varepsilon_v}{3} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} - \frac{\varepsilon_v}{3} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} - \frac{\varepsilon_v}{3} \end{bmatrix} \quad (2.12)$$

Note that in the following, the colon symbol $:$ indicates the inner product of two tensors, i.e., the trace of their product. In particular, if $\mathbf{a}$ and $\mathbf{b}$ are two tensors, it yields $\mathbf{a} : \mathbf{b} = \text{tr}(\mathbf{a} \cdot \mathbf{b}) = a_{ij}b_{ji}$. In addition, the symbol $\langle \rangle$ denotes the Macaulay brackets, and if x is a scalar, $\langle x \rangle = \frac{x + |x|}{2}$.

2.3.1.2 Plasticity Theory

With a view to effectively describing the key aspects of soil behaviour, also under monotonic loadings, the elastic models require the integration with the plasticity theory [64, 122, 112], which, in turn, is based on the following assumptions:

1. Coincidence between incremental plastic strain and accumulated stress principal directions;

2. Existence of a yield function dependent on both stress state $\boldsymbol{\sigma}$ and state parameters $\mathbf{k}$, describing hardening or softening, defined as in 2.13:

$$f(\boldsymbol{\sigma}, \mathbf{k}) = 0 \quad (2.13)$$

The latter must be convex and separates the purely elastic stress states, where $f(\boldsymbol{\sigma}, \mathbf{k}) < 0$, from the elasto-plastic stress states $f(\boldsymbol{\sigma}, \mathbf{k}) = 0$. In this case, the stress state is located on the yield surface, and the consistency condition needs to be guaranteed [122] as expressed in 2.14:

$$df(\boldsymbol{\sigma}, \mathbf{k}) = \frac{\partial f(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} : d\boldsymbol{\sigma} + \frac{\partial f(\boldsymbol{\sigma}, \mathbf{k})}{\partial \mathbf{k}} d\mathbf{k} = 0 \quad (2.14)$$

For the present applications, please note that all stress quantities are effective. The condition $f(\boldsymbol{\sigma}, \mathbf{k}) > 0$ corresponds to non-physical states and, therefore, cannot be admissible. Consequently, the outward normal unit tensor to the yield surface can be expressed as in 2.15:

$$\mathbf{n} = \frac{\frac{\partial f(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}}}{\left\| \frac{\partial f(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} \right\|} \quad (2.15)$$

where the denominator indicates the norm of $\frac{\partial f(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}}$.

3. The additive decomposition of strain increments, which can be switched into an elastic and a plastic term, is valid as 2.16:

$$d\boldsymbol{\varepsilon} = d\boldsymbol{\varepsilon}^e + d\boldsymbol{\varepsilon}^p \quad (2.16)$$

In 2.16, $d\boldsymbol{\varepsilon}$, $d\boldsymbol{\varepsilon}^e$ and $d\boldsymbol{\varepsilon}^p$ are the total strain, elastic strain, and plastic strain tensor increments, respectively. Furthermore, the additive decomposition of strains (cf. 2.16) can be written in incremental form by switching the total strain increment into a volumetric term (see 2.17) and a deviatoric term 2.18:

$$d\boldsymbol{\varepsilon}_v = d\boldsymbol{\varepsilon}_v^e + d\boldsymbol{\varepsilon}_v^p \quad (2.17)$$

$$d\mathbf{e} = d\mathbf{e}^e + d\mathbf{e}^p \quad (2.18)$$

Then, the stress rate can be expressed in terms of strain increments through the following hypoelastic relationship 2.19:

$$d\boldsymbol{\sigma} = \mathbf{D}^e(\boldsymbol{\sigma}) d\boldsymbol{\epsilon}^e = \mathbf{D}^e(\boldsymbol{\sigma}) [d\boldsymbol{\epsilon} - d\boldsymbol{\epsilon}^p] \quad (2.19)$$

In 2.19, $\mathbf{D}^e(\boldsymbol{\sigma})$ is the fourth-order elastic tangent stiffness tensor, and it depends on the current stress state.

4. Existence of a flow rule. The latter describes the plastic strain evolution and can be expressed by the following relation 2.20:

$$d\boldsymbol{\epsilon}^p = \Lambda \frac{\partial g(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} \quad (2.20)$$

where $g(\boldsymbol{\sigma}, \mathbf{k})$ represents the plastic potential function, Λ is a scalar value called the plastic multiplier, and $\mathbf{k}$ indicates the state variables, as in 2.14. The plastic multiplier determines the plastic strain increment modulus, while $\frac{\partial g(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}}$ is the outward normal vector to the plastic potential surface and defines its components. Then, the outward normal unit tensor to the plastic potential function $\mathbf{n}_g$ can be expressed as in 2.21:

$$\mathbf{n}_g = \frac{\frac{\partial g(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}}}{\left\| \frac{\partial g(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} \right\|} \quad (2.21)$$

Note that the state parameters are constant for perfect plasticity, while they vary with the plastic strain otherwise. It is worth noting that some simplifying assumptions can be considered, such as the associated flow hypothesis, which stems from the condition $f(\boldsymbol{\sigma}, \mathbf{k}) = g(\boldsymbol{\sigma}, \mathbf{k})$. This assumption implies a coincidence between the yield surface and plastic potential functions. On the contrary, the flow is said to be non-associated.

5. Existence of a hardening rule, which describes the evolution of the state parameters and, consequently, of the yield surface as a function of the plastic strain increments. The hardening law can be expressed as 2.22:

$$d\mathbf{k}(\boldsymbol{\sigma}, \mathbf{k}) = \Lambda \mathbf{h}(\boldsymbol{\sigma}, \mathbf{k}) \quad (2.22)$$

and $\mathbf{h}(\boldsymbol{\sigma}, \mathbf{k})$ depends on the current stress and state parameters. From a mathematical point of view, the hardening law describes the evolution of the yield surface, which can either translate and/or expand. It defines the post-yielding behaviour of the constitutive

model. Accordingly, the hardening can be either isotropic or kinematic. The kinematic hardening (or softening) takes place when a translation of the hardening surface in the stress space occurs, and any variation of its dimension or shape is forbidden, while the isotropic hardening (or softening) occurs when the yield surface expands (or contracts) without translating and keeping its shape, i.e., maintaining its center and normal unit vectors fixed. When both these behaviours take place, the model is said to exhibit mixed hardening. All the models considered in this study provide a kinematic hardening rule.

The kinematic hardening was first proposed by Prager [122], who developed the first simple kinematic hardening rule. To this aim, the yield surface can be described as in 2.23:

$$f(\boldsymbol{\sigma} - \boldsymbol{\alpha}, \mathbf{k}) = m \quad (2.23)$$

where $\boldsymbol{\alpha}$ is the anisotropy back-stress tensor that expresses the evolution of the center of the yield surface, and m is a material constant representing its size, which remains unchanged.

With reference to the plastic multiplier Λ , it must satisfy the so-called Kuhn–Tucker conditions, enumerated in 2.24:

$$\Lambda \geq 0; \quad f(\boldsymbol{\sigma}, \mathbf{k}) \leq 0; \quad \Lambda f(\boldsymbol{\sigma}, \mathbf{k}) = 0 \quad (2.24)$$

This means that plastic strain can develop only when the current stress state lies on the yield surface. The concept expressed in 2.25 implies that the plastic multiplier Λ is equal to zero for elastic states, while it becomes active when the yield function reaches zero (i.e., the stress state lies on the yield surface). Once activated, the plastic multiplier governs the development of plastic strain.

The basic assumptions of the plasticity theory are graphically shown in Figure 2.4. By applying the consistency condition in 2.14, together with the expressions of stress increments 2.16 and the flow rule 2.20, the plastic multiplier can be expressed as 2.25:

$$\Lambda = \frac{\frac{\partial f(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} : \mathbf{D}^e : d\boldsymbol{\varepsilon}}{\frac{\partial f(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} : \mathbf{D}^e : \frac{\partial g(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} + K_p} \quad (2.25)$$

K_p is called the *plastic modulus*, and in terms of internal variables, by applying the consistency condition, it can be expressed as in 2.26:

$$K_p = -\frac{1}{\Lambda} \frac{\partial f(\boldsymbol{\sigma}, \mathbf{k})}{\partial \mathbf{k}} : d\mathbf{k} \quad (2.26)$$

By substituting the expression of the plastic multiplier into 2.18, the so-called *elasto-*

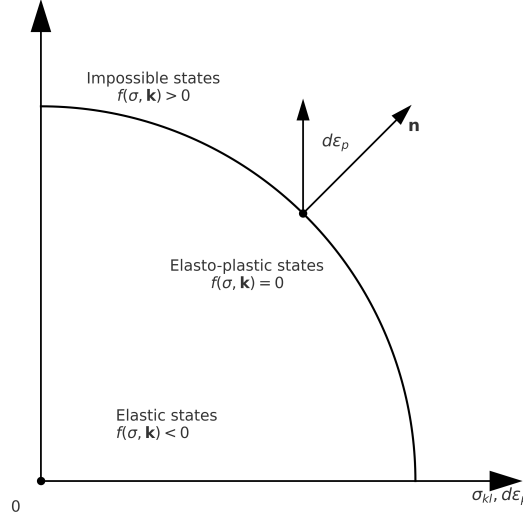


Fig. 2.4 Basic assumptions of the plasticity theory

plastic stiffness matrix, which is a fourth-order tensor, can be defined as 2.27:

$$\mathbf{D}^{ep} = \mathbf{D}^e - \frac{\left(\mathbf{D}^e \frac{\partial g(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} \right) \left(\frac{\partial f(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} \mathbf{D}^e \right)}{\frac{\partial f(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} : \mathbf{D}^e : \frac{\partial g(\boldsymbol{\sigma}, \mathbf{k})}{\partial \boldsymbol{\sigma}} + K_p} \quad (2.27)$$

The evaluation of the elasto-plastic stiffness matrix is crucial in the implementation of constitutive models within numerical codes; therefore, further details are provided in §3. The plasticity theory assumes a clear separation between elastic and elasto-plastic states. However, for soils, it has been observed that irreversible strains can arise even under low stress levels, particularly in the case of cyclic excitations. Consequently, soil nonlinearity can develop even at small stress levels. In this sense, the generation of excess pore-water pressure under undrained conditions plays a key role. For these reasons, the classical elasto-plasticity theory is not adequate to reproduce the main features of soil response under cyclic loadings. To successfully capture these aspects,

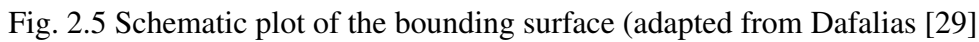
classical plasticity must be integrated with additional concepts, such as the *bounding surface theory* and the *critical state soil mechanics* (CSSM) framework.

2.3.2 Bounding Surface

With the adoption of a single yield surface, the comparison between numerical and experimental results shows a wider elastic domain compared to that obtained from laboratory tests [162]. For this reason, to overcome this limitation, one of the most widespread approaches is represented by the bounding surface theory [28, 82, 29]. The presentation of the concept is detailed in Dafalias and Popov [28], while the mathematical description of the problem is reported in Dafalias [29]. However, several authors have contributed to the development of this theoretical framework (see [124, 123, 108, 109, 62, 63, 1, 54, 129]). The starting hypothesis of this approach states that, considering a stress state inside or on the bounding surface, a mapping rule will associate an “image” stress state $\boldsymbol{\sigma}^b$ on the bounding surface and the distance between the above-mentioned stress states is a measure of the plastic response of the system. More specifically, the theory assumes that the post-yielding behaviour, in terms of plastic modulus, is connected to the distance between the current stress state and the “image” stress state on the bounding surface. More the latter is high more the material behaves as stiff [162]. The bounding surface divides the physical states from impossible states, but some modifications have been made on this point, allowing the bounding surface to be momentarily crossed (see Andrianopoulos et al. [4]). A bounding surface is defined similarly to a yield surface, as follows (cf. Potts and Zdravković, [164]):

$$f(\boldsymbol{\sigma}^b, \mathbf{k}) = 0 \quad (2.28)$$

A plastic potential, a plastic modulus, and a hardening law are defined for the bounding surface. A unit tensor $\mathbf{n}_L$ representing the loading direction can be defined. The above-mentioned quantities are reported in Figure 2.5. As it is clearly shown in Figure 2.5, if the stress state of the soil is on the bounding surface, the latter behaves as a yield surface. Differently from the latter, plastic strain can take place whether the stress state is inside the bounding surface but on the yield surface. During unloading, the soil behaves elastically only if a finite elastic domain exists. Otherwise, in models with a vanishing elastic region, plastic strains occur immediately, including at load reversal. A continuous non-invertible mapping rule is required to relate the current stress state $\boldsymbol{\sigma}$ to the image stress state $\boldsymbol{\sigma}^b$ on the bounding surface and to estimate the distance between them. The distance between these two states can be calculated as in 2.29:



When such a relationship is determined, the loading index is the gradient of the bounding surface and depends on δ . In this way, the distance between the current and the image stress states influences the hardening response of the soil. Analyzing in more detail the situation shown in Figure 2.5, the soil element is loaded in its loading direction. During the loading history, the stress state moves from the initial condition $\boldsymbol{\sigma}_0$ on the yield surface towards the stress state $\boldsymbol{\sigma}$. A first yield surface is defined homothetically from the bounding surface and, consequently, it has the same shape as the bounding surface but scaled down, intersecting the stress state before reaching $\boldsymbol{\sigma}_0$. When the stress state of the soil reaches $\boldsymbol{\sigma}$, if unloaded, the behaviour is purely elastic, while the yield surface evolves based on its hardening rule otherwise. The condition illustrated in Figure 2.5 implicitly shows the so-called *radial mapping rule*, which is the simplest conceived mapping rule. In fact, the latter takes place when the “image” stress state lies on the intersection between the straight line passing through the stress state and the origin of the axes and the bounding surface. It is worth noting that this rule has been shown to satisfactorily reproduce many stress paths in several applications [164].

Furthermore, the homothetism between the bounding surface and the yield surface ensures identical gradients. The evolution of the yield surface hardening is governed by the distance

between the current stress point and the corresponding image stress point on the bounding surface, together with the plastic modulus defined on the bounding surface [164].

2.3.3 Advance Constitutive Models

The constitutive framework developed by Manzari and Dafalias (1997) represents a major advance in the modelling of saturated sand behaviour, as it successfully combined the principles of critical state soil mechanics with the bounding surface plasticity theory. This formulation, later refined in models such as Manzari and Dafalias [30] and PM4SAND [17], was conceived to reproduce both monotonic and cyclic responses of sands under drained and undrained conditions through a single set of parameters. By introducing a multi-surface plasticity approach (including critical, dilatancy, and bounding surfaces) and linking the evolution of state variables to the state parameter ψ , these models achieved an accurate description of soil behaviour across a wide range of densities and stress paths. The bounding surface concept, in particular, proved essential to capture stiffness degradation, stress-history dependency, and hysteretic response during cyclic loading, while maintaining consistency with critical state theory. Building upon the success of these elasto-plastic formulations for saturated soils, similar theoretical frameworks have been extended to the partially saturated domain, where the coexistence of air and water phases introduces additional hydro-mechanical coupling and complexity in the constitutive response. The development of constitutive laws for partially saturated soils under monotonic loads is based on elasto-plasticity theory [56]. However, for simulating cyclic behaviour, additional features must be incorporated, such as isotropic and kinematic hardening. The bounding surface approach has proven highly valuable for advancements in this regard [92], enabling the accurate representation of hysteretic response even when accounting for stress history effects, gradual stiffness degradation during loading, preservation of small-strain stiffness, and early onset of plastic strains [55, 72]. Moreover, the bounding surface concept provides simplicity and computational efficiency in cyclic numerical modelling, even from an implementation standpoint. Considering the various approaches for the **FCV** available in the literature ([65]; [16]; [149], [150]), Gens [57] proposed classifying a constitutive model into three types (i.e., class I, II, and III). Class I is designated for those constitutive models for which $u_1 = 0$, thus they are formulated in terms of net stresses. A class II constitutive model is considered when $u_1(s)$ is dependent only on suction, thus implicitly accounting for the effect of suction in shear strength. Finally, a constitutive model is considered class III when $u_1(s, S_r)$ depends on both suction and degree of saturation and is linked with the **SCV**. Traditionally, classes II and III are referred to as the effective approach.

While classes I and II cannot accurately capture the correct transition from saturated to unsaturated stages, and vice versa, and may encounter some difficulties related to hydraulic hysteresis, constitutive models under class III can address those issues.

For a more comprehensive overview of the evolution of constitutive models, readers are referred to Perez et al. [120], where several formulations specifically designed to simulate cyclic loading conditions are reviewed, including those proposed by Yang et al. [155], Khalili et al. [71], Bian and Shahrour [11], Pedroso and Farias [119], Liu and Muraleetharan [87], Zhou et al. [170], Zhou and Ng [168], Komolvilas and Kikumoto [77], Xiong et al. [154], Zhou et al. [171], Li et al. [85], Ghorbani et al. [61], and Cai et al. [20]. Among these, particular emphasis is placed on the constitutive model proposed by Liu and Muraleetharan [87], as it was adopted in the present work for implementation purposes and will be discussed in detail in a later section.

2.4 Liu and Muraleetharan (2012)

The constitutive model proposed by Liu and Muraleetharan [87] was developed within the framework of bounding surface plasticity and, according to Gens [57], corresponds to Class II of the previously mentioned classification. Drawing upon these concepts, the hydro-mechanical model developed by Liu and Muraleetharan [87] included the concept of bounding surface, guided by the contributions of Manzari and Dafalias [92], Dafalias and Manzari [30] and Taiebat and Dafalias [136]. Consequently, the model formulation adheres to the general principles of classical elasto-plasticity, incorporating a cone-type yield criterion characterized by both isotropic and kinematic hardening. On the other hand, with respect to the elastic behaviour within the model, the theory of hypoelasticity was adopted. Three additional surfaces are defined, namely the dilatancy surface, critical state surface, and bounding surface. They are obtained from Taiebat and Dafalias [136]. The whole framework is reported in Figure (2.6)a in the triaxial space and in Figure (2.6)b in the deviatoric space. Within the encompassing context of the general stress space, the yield surface can be represented by Equation 2.30, while the remaining surfaces are defined through the stress ratios α_θ^b , α_θ^d , and α_θ^c , respectively.

$$f(\boldsymbol{\sigma}', s_c; \boldsymbol{\alpha}, m) = \sqrt{(\mathbf{s} - I\boldsymbol{\alpha}) : (\mathbf{s} - I\boldsymbol{\alpha})} - \sqrt{\frac{2}{3}} m I \sqrt{1 - \left(\frac{I}{I_0}\right)^\beta} = 0 \quad (2.30)$$

$$p' = I = \frac{\sigma'_{11} + \sigma'_{22} + \sigma'_{33}}{3} = \frac{\sigma_{11} + \sigma_{22} + \sigma_{33}}{3} - p_a + n_w s_c, \quad (2.31)$$

where σ_{ij} and σ'_{ij} are the total stress and average skeleton stress tensors, respectively; I = hydrostatic average skeleton stress, in which the matric suction s_c is explicitly included through the term $n_w s_c$; n_w = volumetric water content; p_a = pore air pressure; α = center of the yield surface; m = isotropic hardening parameter, which considers the size of the yield surface defined in the triaxial space; I_0 = initial hydrostatic average skeleton stress; β = model parameter. The axis of the yield surface in the deviatoric space is defined by the

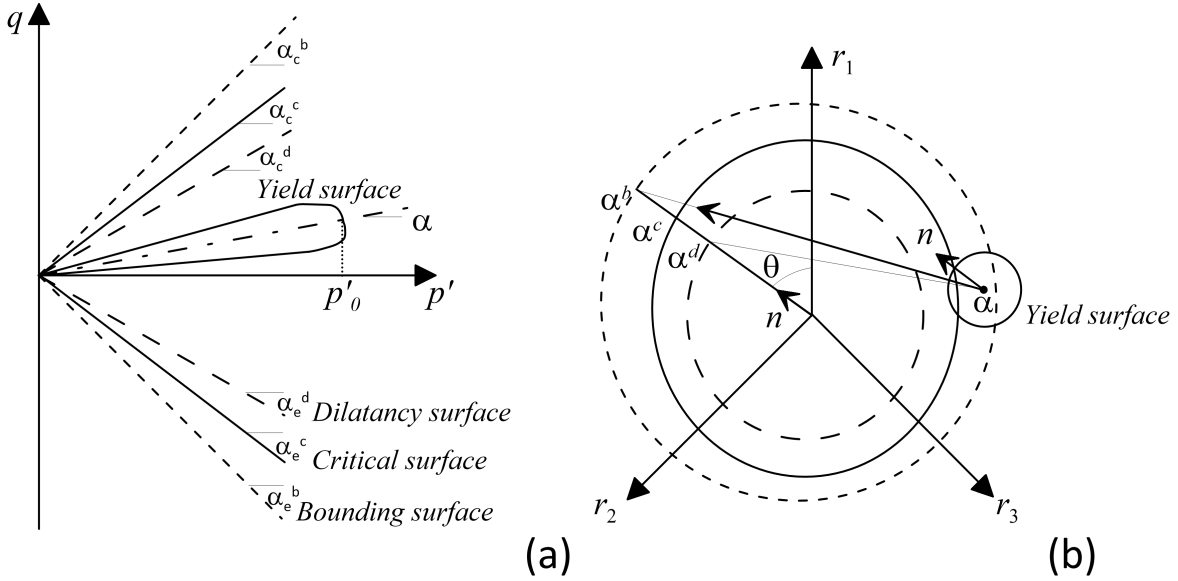


Fig. 2.6 Li and Muraleetharan [87] surfaces model in (a) triaxial space and (b) deviatoric space (adapted from Taiebat and Dafalias [136]).

back-stress ratio α , and the kinematic hardening is consequent to its variation. Notably, the influence of suction on kinematic hardening is confined to its contribution within the average skeleton tensor. Conversely, the isotropic hardening parameter m is not only a function of the stress state but also encapsulates the plastic volumetric strain and the irrecoverable water content. The kinematic and isotropic hardening rules are defined by Equation 2.32 and Equation 2.33, respectively:

$$\dot{\alpha} = h(\alpha_\theta^b - \alpha) = \langle \Lambda \rangle h b \quad (2.32)$$

$$\dot{m} = c_m(1 + e_0)\dot{\epsilon}_v^p + c_v \left(\frac{s_c n_w}{p_{\text{ref}}} \right)^{\bar{\omega}} \dot{n}_w^p \quad (2.33)$$

where Λ is the loading index; c_v , c_m , and $\bar{\omega}$ are model parameters; n_w^p is the plastic volume fraction; $\alpha_\theta^b - \alpha$ denotes the distance between α and its image on the bounding

surface, α_θ^b ; h is a parameter dependent on load reversal; p_{ref} is a reference pressure; ϵ_v^p is the plastic volumetric strain.

The strain tensor of the soil skeleton can be decomposed into hydrostatic and deviatoric components as follows:

$$\dot{\epsilon}^p = \langle \Lambda \rangle \mathbf{R}, \quad \mathbf{R} = \mathbf{n} + \frac{1}{3} D \mathbf{I}, \quad \dot{\epsilon}_q^p = \langle \Lambda \rangle \mathbf{n}, \quad \dot{\epsilon}_v^p = \langle \Lambda \rangle D \quad (2.34)$$

$$D = B(\alpha_\theta^d - \alpha) : \mathbf{n} = B \mathbf{d} : \mathbf{n} \quad (2.35)$$

$$B = B_0 (1 + \langle \mathbf{F} : \mathbf{n} \rangle) \quad (2.36)$$

$$\dot{\mathbf{F}} = -\langle \Lambda \rangle c_f \langle -D \rangle (F_{\text{max}} \mathbf{n} + \mathbf{F}) \quad (2.37)$$

where $\mathbf{R}$ is the direction of $\dot{\epsilon}^p$, consisting of a deviatoric component $\mathbf{n}$ and a volumetric component $(1/3)D\mathbf{I}$; D is the dilatancy coefficient; B_0 and c_f are model parameters, and F_{max} is another model parameter representing the maximum value that the fabric tensor $\|\mathbf{F}\|$ can attain.

Liu and Muraleetharan [87] incorporated the Soil-Water Retention Curves (SWRC) put forth by Liu and Muraleetharan [86] and Muraleetharan et al. [111] in the concept of bounding surface plasticity. The primary $(p)n_w$ and the secondary $(s)n_w$ wetting curves are described by Equation 2.38 and were proposed by Feng and Fredlund [37], neglecting the primary drying curve as the boundary curve due to air entrapment. In the formula, n_{res} is considered as the model parameter, and the capillary elastic and plastic moduli Γ^e (assumed constant) and Γ^p are obtained by Equations 2.39 and 2.40, respectively.

$$(p)n_w = \frac{n_{\text{sat}} + n_{\text{res}} \left(\frac{s_{c0w}}{b_1} \right)^{d_1}}{1 + \left(\frac{s_{c0w}}{b_1} \right)^{d_1}}; \quad (s)n_w = \frac{n_{\text{sat}} + n_{\text{res}} \left(\frac{s_{c0d}}{b_2} \right)^{d_2}}{1 + \left(\frac{s_{c0d}}{b_2} \right)^{d_2}} \quad (2.38)$$

$$\dot{s}_c = \Gamma^e \dot{n}_w^e; \quad \dot{s}_c = \Gamma^p \dot{n}_w^p \quad (2.39)$$

$$\Gamma^p = \Gamma^p(\delta, \delta_{\text{in}}) = \Gamma_0^p \left(1 + \frac{H\delta}{\langle \delta_{\text{in}} - g\delta \rangle} \right) \quad (2.40)$$

Where $n_{\text{sat}}, n_{\text{res}}$ = volumetric and residual water content at zero suction; b_1, d_1, b_2, d_2 = material parameters; δ = distance between the current stress point and the image point on

the bounding surface; δ_{in} = distance between the stress state at the initiation of yielding for each drying/wetting process and its image on the bounding surface; $\dot{s}_c$ = rate of change of suction; $\dot{n}_w^e, \dot{n}_w^p$ = rate of change of recoverable and irrecoverable volume fractions of water, respectively; H = shape parameter.

To explore the coupling between the mechanical and hydraulic behaviour of unsaturated sands or silts, the progression of the bounding suctions on the drying bound s_{c0d} and wetting bound s_{c0w} is expressed as follows:

$$\dot{s}_{c0w} = \Gamma_{vw}^p \dot{\epsilon}_v^p + \Gamma_{0w}^p \dot{n}_w^p; \quad \Gamma_{vw}^p = s_{c0w} \zeta v \quad (2.41)$$

$$\dot{s}_{c0d} = \Gamma_{vd}^p \dot{\epsilon}_v^p + \Gamma_{0d}^p \dot{n}_w^p; \quad \Gamma_{vd}^p = s_{c0d} \zeta v \quad (2.42)$$

where ζ is a material parameter; v is the specific volume; s_{c0w}, s_{c0d} are the suctions on the wetting and drying bounding curves; and $\Gamma_{0w}^p, \Gamma_{0d}^p$ are the capillary plastic moduli on the wetting and drying bounds, respectively.

Liu and Muraleetharan [87] proposed to obtain the loading index Λ by Equation (2.43). Here, $\frac{\partial f}{\partial I}$ is a function of I, β, s (deviatoric stress), I_0 and m ; Γ^p is the capillary plastic modulus; K_s^p and K_m^p are two plastic moduli, with the former representing the classic plastic modulus in the elasto-plasticity theory and the latter being a new plastic modulus that takes into account the hydro-effects on the yield surface; $\dot{s}_c$ establishes the rate of change of the suction.

$$\Lambda = \frac{1}{K_s^p} \left\{ \frac{(s - I\alpha) : \dot{s}}{\sqrt{(s - I\alpha) : (s - I\alpha)}} + \frac{\partial f}{\partial I} \dot{I} - \frac{K_m^p}{\Gamma^p} \dot{s}_c \right\} \quad (2.43)$$

Chapter 3

Implementation of the Constitutive Model in the Constitutive Driver and OpenSees

3.1 Introduction

The constitutive model developed by Li and Muraleetharan [87] has proven particularly promising due to its capability to simulate a wide range of geotechnical problems involving partially saturated soils. Despite its theoretical and numerical potential, the model has not yet been implemented within a widely adopted finite element platform. This limitation has significantly reduced its practical impact and broader applicability in boundary value problems.

In this context, the present work investigates the effectiveness of the model through its implementation in OpenSees. This choice is motivated by two main reasons: i) to broaden the use of the constitutive model by embedding it within a widely adopted, open-source finite element framework, and ii) to expand the capabilities of OpenSees by enabling the solution of a new class of problems involving partially saturated soils.

Prior to its incorporation into OpenSees, the constitutive model was implemented within an in-house Constitutive Driver. This preliminary stage was essential to validate the numerical formulation of the model, assess its robustness, and verify its predictive capabilities under controlled loading conditions representative of laboratory tests. By isolating the constitutive response from the complexities associated with boundary value problems, the driver provides a transparent environment for debugging, verification, and parametric investigation.

This strategy is not new in this work, but it is often adopted within the scientific community dealing with soil plasticity (see, for example, Sloan [134]; Mattson et al. [95]).

Numerical simulations were conducted using the Constitutive Driver in order to reproduce the laboratory tests reported by Liu and Muraleetharan [87].

In addition to reproducing the original formulation, a simplified variant was also developed to facilitate a robust implementation within OpenSees. Specifically, the mechanical backbone of the model is recast within a SANISAND/Manzari–Dafalias stress–ratio framework, while the hydraulic state is described through a Van Genuchten retention law. This implementation, herein referred to as `ManzariDafaliasIsoHard`, retains the essential mechanical features of the original model while avoiding hydraulic internal variables that are difficult to track consistently at the Gauss-point level in general boundary value problems.

The following sections first describe the structure and functionality of the MATLAB-based Constitutive Driver and the numerical integration strategy adopted to reproduce the reference laboratory stress paths. Subsequently, the OpenSees implementation is presented, including the rationale and main features of the simplified `ManzariDafaliasIsoHard` formulation adopted for boundary value problems.

3.2 Integration of Nonlinear Constitutive Equations

The strong non-linearity inherent to bounding surface constitutive models precludes the analytical integration of the stress–strain relationships. Accordingly, within a finite element framework, the stress increment associated with a prescribed strain increment is obtained through numerical integration of the constitutive equations formulated in rate form. These equations encompass the elastic response, the plastic flow rule derived from the plastic potential, and the evolution laws of the internal variables controlling the material behaviour, such as hardening parameters and fabric tensors accounting for induced anisotropy.

Numerical integration schemes can be subdivided into explicit methods and implicit methods. The former methods integrate equations at known stress states with consequent no iteration strictly required to predict final stress and, generally, consistency condition is not assured. This phenomenon is called yield surface drift. For this reason, it is prudent to provide the existence of a simple stress correction in order to assure that the stress state lies on the yield surface [135]. On the contrary, implicit methods compute the solution by considering iterative strategies where all the unknown variables are calculated at the end of the step increment. This guarantees that consistent stress states are automatically furnished in output.

3.2.1 Integration within the Constitutive Driver

The objective of this preliminary stage was to verify the correctness of the constitutive formulation, assess the numerical robustness of the integration algorithm, and reproduce the laboratory stress paths reported by Liu and Muraleetharan [87] under controlled conditions.

Within the Constitutive Driver, the constitutive equations are integrated at the material-point level under prescribed loading paths, without coupling to a global equilibrium problem. This allows direct control over the stress and strain histories, stress updates, and internal variables, and facilitates the identification of potential numerical issues prior to the implementation within a finite element framework.

In the present work, a higher-order explicit Runge–Kutta–Fehlberg scheme with embedded orders five and six (RKF 5(6)) is adopted. This choice is motivated by the strongly nonlinear and path-dependent nature of the constitutive model, which involves coupled hydro-mechanical evolution equations and may exhibit rapid changes in stiffness, particularly under undrained and partially saturated conditions. Increasing the integration order reduces the local truncation error and allows larger effective substeps to be taken while preserving numerical accuracy.

Explicit RKF schemes provide a robust and efficient strategy for integrating advanced soil constitutive models, as they avoid local iterative stress-correction procedures while maintaining accuracy through adaptive substepping and embedded error control [138]. In particular, higher-order RKF methods have been shown to significantly reduce the total number of constitutive function evaluations compared to lower-order explicit schemes, resulting in improved computational efficiency at the material-point level [93].

Based on these considerations, the following subsection describes the numerical integration strategy adopted in the Constitutive Driver, with particular emphasis on the formulation of the Runge–Kutta–Fehlberg RKF 5(6) algorithm, its adaptive substepping procedure, and the associated error control mechanism.

3.2.1.1 Runge–Kutta–Fehlberg (RKF) with Substepping

The Runge–Kutta–Fehlberg (RKF) scheme is an *explicit, adaptive* time-integration algorithm that provides two embedded Runge–Kutta solutions of different order within each integration substep. The difference between these two solutions is used to estimate the local truncation error, which in turn enables automatic control of the substep size.

In the present work, the *Dormand–Prince* Runge–Kutta–Fehlberg pair with embedded orders five and six, denoted as RKF 5(6), is adopted. This scheme offers an excellent compromise between accuracy and computational efficiency and exhibits favourable stability

properties when applied to strongly nonlinear constitutive rate equations, such as those governing the coupled hydro-mechanical response of partially saturated soils.

In the RKF(5,6) formulation, the internal state vector $\mathbf{u}$, which represents stresses and internal variables, evolves according to the ordinary differential equation $\dot{\mathbf{u}} = \mathbf{f}(\mathbf{u})$. Its value at the end of the time step Δt is obtained by computing a series of intermediate stages $\mathbf{k}_i$.

$$\mathbf{k}_i = \mathbf{f}\left(\mathbf{u}^n + \Delta t \sum_{j=1}^{i-1} a_{ij} \mathbf{k}_j\right), \quad i = 1, \dots, 7. \quad (3.1)$$

The coefficients a_{ij} , $b_i^{(5)}$ and $b_i^{(6)}$ correspond to the *Dormand–Prince tableau* (see, e.g., 134; 138), where the fifth-order solution $\tilde{\mathbf{u}}^{n+1}$ and the sixth-order solution $\mathbf{u}^{n+1}$ are given by 3.2 and 3.3.

$$\tilde{\mathbf{u}}^{n+1} = \mathbf{u}^n + \Delta t \sum_{i=1}^7 b_i^{(5)} \mathbf{k}_i, \quad (3.2)$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \sum_{i=1}^7 b_i^{(6)} \mathbf{k}_i. \quad (3.3)$$

A key feature of the Runge–Kutta–Fehlberg integration strategy is the explicit introduction of adaptivity in the integration process. Rather than prescribing a fixed integration increment, the RKF scheme dynamically adjusts the substep size in response to the local behaviour of the constitutive equations, allowing smaller substeps to be adopted in highly nonlinear regimes and larger substeps when the response is smooth.

The local truncation error E is estimated from the difference between the two embedded solutions:

$$E = \frac{\|\mathbf{u}^{n+1} - \tilde{\mathbf{u}}^{n+1}\|}{\|\mathbf{u}^{n+1}\| + \varepsilon_m}, \quad (3.4)$$

Where ε_m is a small regularization parameter (machine tolerance). If the computed error satisfies $E \leq \text{STOL}$, the step is accepted; otherwise, it is rejected and recomputed with a smaller increment.

The adaptive nature of the Runge–Kutta–Fehlberg scheme is governed by the local error estimate, which is used to automatically adjust the size of the subsequent integration substep according to Eq. (3.5).

$$\beta = 0.8 \left(\frac{\text{STOL}}{E} \right)^{1/(p+1)}, \quad (3.5)$$

where p denotes the order of the lower-order method in the embedded Runge–Kutta pair. For the RKF(5,6) scheme, $p = 5$. The factor 0.8 acts as a safety coefficient, while β is limited to prevent abrupt changes in step size, typically $\beta \in [0.1, 2.0]$. For highly nonlinear constitutive behaviour, a more conservative upper bound $\beta \leq 0.5$ is often recommended (see, e.g., 3).

Within each *accepted* RKF substep, the RKF scheme is used exclusively to control the integration increment (adaptive substepping) and advance the state variables. Yielding is detected separately through an *elastic trial* evaluation. In particular, a trial stress state $\boldsymbol{\sigma}_{n+1}^{\text{tr}}$ is computed by assuming purely elastic behaviour over the accepted substep increment. The yield function is then evaluated at the trial state to determine whether plasticity is activated. If $f(\boldsymbol{\sigma}_{n+1}^{\text{tr}}, \boldsymbol{\alpha}) \leq \text{FTOL}$, the substep is considered elastic and the trial state is accepted. Otherwise, the stress path is assumed to cross the yield surface within the substep, and the elastoplastic transition is located by solving the intersection problem

$$f(\boldsymbol{\sigma}_n + \alpha(\boldsymbol{\sigma}_{n+1}^{\text{tr}} - \boldsymbol{\sigma}_n), \boldsymbol{\alpha}) = 0, \quad \alpha \in [0, 1], \quad (3.6)$$

where α is computed using a robust derivative-free root-finding algorithm (e.g. the Pegasus method; 33). This procedure identifies the onset of plasticity within the substep and mitigates numerical drift with respect to the yield surface.

Special care is required during unloading phases, particularly when the angle between the plastic flow direction $\partial f / \partial \boldsymbol{\sigma}$ and the trial stress increment exceeds 90° . In such situations, the trial stress path may cross the yield surface more than once, leading to potential inaccuracies if a standard return-mapping scheme is adopted. The interested reader is referred to the seminal work of Sloan [135], which provides a detailed discussion of loading–unloading criteria and robust numerical strategies to correctly handle these cases.

The interested reader can refer to Dowell and Jarratt [33] for further details, while for its implementation in the integration of the elasto-plastic matrix for critical state constitutive models, Sloan et al. [135] provide the full description of the algorithm.

However, at the end of each step, the drift (i.e., the divergence of the calculated state of stress from the yield surface) may take place and a correction of the response to the yield surface is needed. This can be made on the basis of Sloan et al. [135], where the yield surface expression may be expanded in Taylor series and, on the basis of the assigned increment, an unknown multiplier can be evaluated iteratively so that consistent response can be obtained.

3.2.2 Constitutive integration in OpenSees

In OpenSees, the constitutive integration scheme is generally embedded within the implementation of each `NDMaterial` class. Depending on the material formulation, the stress

update may be performed using either explicit or implicit integration strategies, which can be hard-coded within the material or, in some cases, exposed to the user as optional input parameters. As a result, the choice of the integration scheme remains material-specific and is not controlled at the finite element or global analysis level.

Within this framework, the implementation of advanced constitutive models typically requires the explicit coding of the stress-update algorithm inside the corresponding `NDMaterial` class. The selected integration scheme is then invoked at each Gauss integration point during the global equilibrium iterations, ensuring consistency between the local constitutive response and the global finite element solution. This design allows different materials to adopt distinct integration strategies while preserving a unified solution architecture within OpenSees.

3.3 Constitutive Driver

The Constitutive Driver is developed following the linearized integration framework originally proposed by Bardet and Choucair [9]. This framework was conceived as a general numerical tool to simulate the stress–strain response of constitutive models under laboratory-type loading conditions by directly integrating rate-type or incremental constitutive equations at the material point level, without resorting to the solution of boundary value problems.

The fundamental idea of the formulation is to represent the loading conditions of laboratory experiments through linearized constraints imposed on stress and strain increments. These constraints are expressed in terms of constant matrices within each loading increment and are coupled with the constitutive relations governing the material response. As a result, the evolution of stress and strain during a prescribed test is governed by the linear system of ordinary differential equations given by Eq. (3.7).

$$\mathbf{S} \dot{\boldsymbol{\sigma}} + \mathbf{E} \dot{\boldsymbol{\epsilon}} = \dot{\mathbf{V}} \quad (3.7)$$

where $\mathbf{S}$ and $\mathbf{E}$ are the constraint matrices associated with stress and strain rates, respectively, and $\dot{\mathbf{V}}$ is the rate of the loading variable driving the experiment.

Equation (3.7) provides a unified mathematical representation of laboratory loading conditions and constitutes the core of the Constitutive Driver. By appropriate definition of the matrices $\mathbf{S}$, $\mathbf{E}$, and $\mathbf{V}$, different types of laboratory tests can be reproduced, including stress-controlled, strain-controlled, and mixed-control experiments. Moreover, since these matrices are assumed constant within each loading increment but may vary between successive increments, complex and nonlinear loading paths can be constructed in a piecewise-linear manner.

For nonlinear constitutive models, the system defined by Eq. (3.7) is integrated numerically. In the present work, the incremental stress update is performed using the refined explicit integration scheme proposed by Sloan et al. [135], which improves numerical robustness and accuracy compared to classical forward Euler methods, particularly in the presence of elastoplastic yielding and complex loading paths.

To represent unsaturated soil behaviour, two constitutive extensions were formulated within the proposed framework. Although both formulations were fully developed at the theoretical level, only the first extension was implemented and employed in the numerical simulations presented in this work, while the second is reported for completeness and future developments.

The first extension follows the work of Buscarnera and Nova [19] and is based on an enlargement of the constitutive stiffness matrix $\mathbf{D}_{\text{ext}}$, with the incremental response expressed in terms of effective stress and degree of saturation. This formulation adopts the work-conjugated stress and strain variables proposed by Houlsby [65], allowing for a consistent hydro-mechanical coupling under unsaturated conditions.

Since the constitutive model proposed by Liu and Muraleetharan [87] is formulated in terms of volumetric water content n_w and average skeleton stress σ''_{ij} , the constitutive stiffness matrix was accordingly adapted and expressed as:

$$\begin{Bmatrix} \dot{\sigma}''_{ij} \\ \dot{s} \end{Bmatrix} = \begin{bmatrix} \mathbf{D}_{\sigma\sigma} & \mathbf{D}_{\sigma w} \\ \mathbf{D}_{w\sigma} & \mathbf{D}_{ww} \end{bmatrix} \begin{Bmatrix} \dot{\epsilon}_{ij} \\ \dot{n}_w \end{Bmatrix} = \mathbf{D}_{\text{ext}} \dot{\mathbf{E}} \quad (3.8)$$

In Equation (3.8), $\mathbf{D}_{\sigma\sigma}$ represents the stiffness matrix of the solid. The vectors $\mathbf{D}_{\sigma w}$ and $\mathbf{D}_{w\sigma}$ correspond to the coupling terms, which capture the contributions of hydraulic coupling to the mechanical behavior of the soil and mechanical coupling to the hydraulic behavior of the unsaturated material, respectively. $\mathbf{D}_{ww}$ is a scalar function that links the suction rate to the rate of change of the volumetric water content. This relationship is fundamentally associated with the water retention curve.

The second extension, following Cattaneo's work [22], represents a more comprehensive formulation, in which the set of controlling and dependent variables was expressed in terms of pore water pressure u_w , pore air pressure u_a and the corresponding water m_w and air m_a masses. Accordingly, Equation 3.8 was modified as shown below. This extension required corresponding changes in the external stiffness matrix $\mathbf{D}_{\text{ext}}$, which are detailed in Section 3.3.2.

$$\begin{Bmatrix} \dot{\boldsymbol{\sigma}}''_{ij} \\ \dot{u}_a \\ \dot{u}_w \end{Bmatrix} = [\mathbf{D}_{\text{ext}}] \cdot \begin{Bmatrix} \dot{\boldsymbol{\epsilon}}_{ij} \\ \dot{m}_a \\ \dot{m}_w \end{Bmatrix} \quad (3.9)$$

3.3.1 Constraints

Since the constitutive model of Liu and Muraleetharan [87] is based on a variation of Bishop's effective stress principle, denominated as intergranular stress, certain modifications were required to regulate the conditions for stress integration. Accordingly, for the first extension, the stress increment was adjusted as follows:

$$\dot{\boldsymbol{\sigma}}^{total} = \dot{\boldsymbol{\sigma}}''_{ij} - n_w \dot{s} - \dot{n}_w s \quad (3.10)$$

For the second one, a more detailed work is needed, and starting from Equation 3.10, we can express it in terms of degree of saturation, pore water and air pressures as represented in Equation 3.11.

$$\dot{\boldsymbol{\sigma}}_{ij} = \dot{\boldsymbol{\sigma}}_{ij}^{total} - \frac{\dot{\boldsymbol{\epsilon}}_v}{1+e} \cdot S_r \cdot (u_a - u_w) + n \cdot \dot{S}_r \cdot (u_a - u_w) + n_w \cdot (\dot{u}_a - \dot{u}_w) \quad (3.11)$$

$$n_w = n \cdot S_r \quad \text{and} \quad S_r = \frac{m_w \rho_a}{m_w \rho_a + m_a \rho_w} \quad (3.12)$$

Where the time derivative of the saturation ratio S_r is given by:

$$\dot{S}_r = \frac{(\dot{m}_w \rho_a + m_w \dot{\rho}_a)(m_w \rho_a + m_a \rho_w) - m_w \rho_a (\dot{m}_w \rho_a + \dot{m}_a \rho_w + m_w \dot{\rho}_a)}{(m_w \rho_a + m_a \rho_w)^2} \quad (3.13)$$

Assuming the air density changes with pore air pressure as Equation 3.14, and expanding the numerator of Equation 3.13, we have Equation 3.15. So, Equation 3.12 is modified as Equation 3.16.

$$\dot{\rho}_a = \frac{\rho_a}{u_a + p_{\text{atm}}} \dot{u}_a, \quad (3.14)$$

$$\dot{S}_r = \frac{\rho_a m_a \rho_w}{(m_w \rho_a + m_a \rho_w)^2} \cdot \dot{m}_w + \frac{m_w \rho_a m_a \rho_w}{(u_a + p_{\text{atm}})(m_w \rho_a + m_a \rho_w)^2} \cdot \dot{u}_a - \frac{m_w \rho_a \rho_w}{(m_w \rho_a + m_a \rho_w)^2} \cdot \dot{m}_a \quad (3.15)$$

$$\begin{aligned}
 \dot{\boldsymbol{\sigma}}_{ij}^{total} = & \dot{\boldsymbol{\sigma}}_{ij}'' + \frac{\dot{\boldsymbol{\epsilon}}_v}{1+e} S_r (u_a - u_w) \\
 & - n \left[\frac{\rho_a m_a \rho_w}{(m_w \rho_a + m_a \rho_w)^2} \cdot \dot{m}_w + \frac{m_w \rho_a \rho_w}{(m_w \rho_a + m_a \rho_w)^2} \cdot \dot{m}_a \right. \\
 & \quad \left. + \frac{m_w \rho_a m_a \rho_w}{(u_a + p_{atm})(m_w \rho_a + m_a \rho_w)^2} \cdot \dot{u}_a \right] (u_a - u_w) \\
 & - n S_r \dot{u}_a + n S_r \dot{u}_w
 \end{aligned} \tag{3.16}$$

In light of these modifications, the constraints matrices were revised. So, this means that the mechanical part is handled using the blocks $\mathbf{S}$ and $\mathbf{E}$ as defined by Bardet and Choucar [9], and the hydraulic part is linear and decoupled, and determine whether pressure or mass is imposed (depending on whether $\Gamma_w = 1$, $\Sigma_w = 0$, or vice versa). Consequently, Equation (3.7) was updated as follows:

$$\mathbf{\Sigma} \Delta \mathbf{s} + \mathbf{\Gamma} \Delta \mathbf{c} = \Delta \mathbf{V} \tag{3.17}$$

Where:

Controlling variables: $\mathbf{c} = [\boldsymbol{\epsilon}, m_a, m_w]^T$; **Dependent variables:** $\mathbf{s} = [\boldsymbol{\sigma}, u_w, u_a]^T$

$$\mathbf{\Sigma} = \begin{bmatrix} \mathbf{S} & 0 & 0 \\ 0 & \Sigma_a & 0 \\ 0 & 0 & \Sigma_w \end{bmatrix}, \quad \mathbf{\Gamma} = \begin{bmatrix} \mathbf{E} & 0 & 0 \\ 0 & \Gamma_a & 0 \\ 0 & 0 & \Gamma_w \end{bmatrix}$$

3.3.2 Extension

Since the controlling and dependent variables are extended, some modifications must be made to the matrix D_{ext} , starting from the plastic multiplier obtained in Equation (2.39). Alternatively, it can be expressed in terms of pore pressures as:

$$\Lambda = \frac{1}{H - H_c} \left[\frac{\partial f}{\partial \boldsymbol{\sigma}} \cdot \mathbf{C} \cdot \dot{\boldsymbol{\epsilon}} + \frac{\partial f}{\partial m} \cdot \frac{\partial m}{\partial n_w^p} \cdot \frac{1}{\Gamma^p} \cdot \dot{u}_a - \frac{\partial f}{\partial m} \cdot \frac{\partial m}{\partial n_w^p} \cdot \frac{1}{\Gamma^p} \cdot \dot{u}_w \right] \tag{3.18}$$

This can be decomposed as:

$$\Lambda = \Lambda_m + \Lambda_a + \Lambda_w \tag{3.19}$$

Using the constitutive law, we obtain:

$$\begin{aligned}\dot{\boldsymbol{\sigma}}'' &= \mathbf{C} \left[\dot{\boldsymbol{\epsilon}} - (\Lambda_m + \Lambda_a + \Lambda_w) \frac{\partial g}{\partial \boldsymbol{\sigma}} \right] \\ \dot{\boldsymbol{\sigma}}'' &= \mathbf{C}^{ep} \dot{\boldsymbol{\epsilon}} - \mathbf{C} \Lambda_a \frac{\partial g}{\partial \boldsymbol{\sigma}} - \mathbf{C} \Lambda_w \frac{\partial g}{\partial \boldsymbol{\sigma}}\end{aligned}\quad (3.20)$$

Expanding the terms explicitly:

$$\begin{aligned}\dot{\boldsymbol{\sigma}}'' &= \underbrace{\mathbf{C}^{ep}}_{D_{\sigma\sigma}} \dot{\boldsymbol{\epsilon}} - \underbrace{\mathbf{C} \left(\frac{\partial g}{\partial \boldsymbol{\sigma}} \right) \frac{1}{H - H_c} \frac{\partial f}{\partial m} \frac{\partial m}{\partial n_w^p} \frac{1}{\Gamma^p}}_{D_{\sigma a}} \dot{u}_a \\ &\quad + \underbrace{\mathbf{C} \left(\frac{\partial g}{\partial \boldsymbol{\sigma}} \right) \frac{1}{H - H_c} \frac{\partial f}{\partial m} \frac{\partial m}{\partial n_w^p} \frac{1}{\Gamma^p}}_{D_{\sigma w}} \dot{u}_w \\ \dot{\boldsymbol{\sigma}}'' &= D_{\sigma\sigma} \dot{\boldsymbol{\epsilon}} + D_{\sigma a} \dot{u}_a + D_{\sigma w} \dot{u}_w\end{aligned}\quad (3.21)$$

For the hydraulic component, the scanning curve of the Soil-Water Retention Curve (SWRC) was differentiated, and the resulting expression is defined as follows:

$$\dot{n}_w = \left(\frac{1}{\Gamma^e} + \frac{1}{\Gamma^p} \right) (\dot{u}_a - \dot{u}_w) \quad (3.22)$$

Taking into account that the water mass balance is defined as $m_w = \rho_w * n_w$, and assuming a constant water density ρ_w , we differentiate the expression with respect to time and obtain Equation (3.23).

$$\dot{m}_w = \rho_w * \dot{n}_w \quad (3.23)$$

Replacing (3.22) in the water mass balance Equation (3.23) we have:

$$\dot{m}_w = \underbrace{\rho_w \left(\frac{1}{\Gamma^e} + \frac{1}{\Gamma^p} \right)}_{D_{wa}} (\dot{u}_a) - \underbrace{\rho_w \left(\frac{1}{\Gamma^e} + \frac{1}{\Gamma^p} \right)}_{D_{ww}} (\dot{u}_w) \quad (3.24)$$

The same procedure applied to the water mass balance is similarly applied to the air mass balance, defined as $m_a = \rho_a V_a$. Its corresponding differential expression is given by:

$$\dot{m}_a = \rho_a \dot{V}_a = \rho_a \left(\frac{\dot{V}_a}{V} \right) = (\rho_a (\dot{n} - \dot{n}_w)) = \dot{\rho}_a (n - n_w) + \rho_a (\dot{n} - \dot{n}_w) \quad (3.25)$$

Replacing (3.25) in the air mass balance we have:

$$\dot{m}_a = \underbrace{\left[\frac{\rho_a}{u_a + p_{\text{atm}}} (n - n_w) - \rho_a \left(\frac{1}{\Gamma_e} + \frac{1}{\Gamma_p} \right) \right]}_{D_{aa}} \cdot \dot{u}_a + \underbrace{\rho_a \left(\frac{1}{\Gamma_e} + \frac{1}{\Gamma_p} \right)}_{D_{aw}} \cdot \dot{u}_w - \underbrace{\rho_a \cdot \frac{1}{1+e}}_{D_{a\sigma}} \cdot \dot{\epsilon}_v \quad (3.26)$$

So, restructuring the matrix as an initial step, we need $\dot{u}_a$ and $\dot{u}_w$ as a function of $\dot{m}_a$ and $\dot{m}_w$, so we inverse the equations (3.24) and (3.25) as following:

$$\begin{bmatrix} \dot{m}_a \\ \dot{m}_w \end{bmatrix} = \begin{bmatrix} D_{aa} & D_{aw} \\ D_{wa} & D_{ww} \end{bmatrix} \begin{bmatrix} \dot{u}_a \\ \dot{u}_w \end{bmatrix} + \begin{bmatrix} D_{a\sigma} \\ 0 \end{bmatrix} \dot{\epsilon} : I \quad (3.27)$$

We invert the matrix as:

$$\mathbf{D}^{-1} = \frac{1}{J} \begin{bmatrix} D_{ww} & -D_{aw} \\ -D_{wa} & D_{aa} \end{bmatrix}, \quad \text{where } J = D_{aa}D_{ww} - D_{aw}D_{wa} \quad (3.28)$$

Then:

$$\begin{bmatrix} \dot{u}_a \\ \dot{u}_w \end{bmatrix} = \mathbf{D}^{-1} \left(\begin{bmatrix} \dot{m}_a \\ \dot{m}_w \end{bmatrix} - \begin{bmatrix} D_{a\sigma} \\ 0 \end{bmatrix} \dot{\epsilon} : I \right) \quad (3.29)$$

$$\begin{aligned} \dot{u}_a &= \frac{1}{J} [D_{ww} \cdot (\dot{m}_a - D_{a\sigma} \dot{\epsilon} : I) - D_{aw} \cdot \dot{m}_w] \\ &= -\frac{D_{ww}D_{a\sigma}}{J} \dot{\epsilon} : I + \frac{D_{ww}}{J} \dot{m}_a - \frac{D_{aw}}{J} \dot{m}_w \end{aligned} \quad (3.30)$$

$$\begin{aligned} \dot{u}_w &= \frac{1}{J} [-D_{wa} \cdot (\dot{m}_a - D_{a\sigma} \dot{\epsilon} : I) + D_{aa} \cdot \dot{m}_w] \\ &= \frac{D_{wa}D_{a\sigma}}{J} \dot{\epsilon} : I - \frac{D_{wa}}{J} \dot{m}_a + \frac{D_{aa}}{J} \dot{m}_w \end{aligned} \quad (3.31)$$

$$\begin{aligned} \dot{u}_a &= -\frac{D_{ww}D_{a\sigma}}{J} \dot{\epsilon} : I + \frac{D_{ww}}{J} \dot{m}_a - \frac{D_{aw}}{J} \dot{m}_w \\ \dot{u}_w &= \frac{D_{wa}D_{a\sigma}}{J} \dot{\epsilon} : I - \frac{D_{wa}}{J} \dot{m}_a + \frac{D_{aa}}{J} \dot{m}_w \end{aligned} \quad \text{with } J = D_{aa}D_{ww} - D_{aw}D_{wa} \quad (3.32)$$

Then:

$$\dot{u}_a = \frac{1}{J} [D_{ww} \cdot (\dot{m}_a - D_{a\sigma} \dot{\epsilon} : I) - D_{aw} \cdot \dot{m}_w] = -\frac{D_{ww}D_{a\sigma}}{J} \dot{\epsilon} : I + \frac{D_{ww}}{J} \dot{m}_a - \frac{D_{aw}}{J} \dot{m}_w \quad (3.33)$$

$$\dot{u}_w = \frac{1}{J} [-D_{wa} \cdot (\dot{m}_a - D_{a\sigma} \dot{\epsilon} : I) + D_{aa} \cdot \dot{m}_w] = \frac{D_{wa}D_{a\sigma}}{J} \dot{\epsilon} : I - \frac{D_{wa}}{J} \dot{m}_a + \frac{D_{aa}}{J} \dot{m}_w \quad (3.34)$$

So, combining equations (3.21), (3.33) and (3.34), we obtain the full matrix:

$$\begin{bmatrix} \dot{\boldsymbol{\sigma}}'' \\ \dot{u}_a \\ \dot{u}_w \end{bmatrix} = \begin{bmatrix} D_{\sigma\sigma} & D_{\sigma a} & D_{\sigma w} \\ -\frac{D_{ww}D_{a\sigma}}{J} : \mathbf{I} & \frac{D_{ww}}{J} & -\frac{D_{aw}}{J} \\ \frac{D_{wa}D_{a\sigma}}{J} : \mathbf{I} & -\frac{D_{wa}}{J} & \frac{D_{aa}}{J} \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\epsilon}} \\ \dot{m}_a \\ \dot{m}_w \end{bmatrix} \quad (3.35)$$

$$\text{With } J = D_{aa}D_{ww} - D_{aw}D_{wa}$$

$$\begin{aligned} D_{a\sigma}^* &= -\frac{D_{ww}D_{a\sigma}}{J} : \mathbf{I}, & D_{aa}^* &= \frac{D_{ww}}{J}, & D_{aw}^* &= -\frac{D_{aw}}{J}, \\ D_{w\sigma}^* &= \frac{D_{wa}D_{a\sigma}}{J} : \mathbf{I}, & D_{wa}^* &= -\frac{D_{wa}}{J}, & D_{ww}^* &= \frac{D_{aa}}{J} \end{aligned}$$

Finally:

$$\begin{bmatrix} \dot{\boldsymbol{\sigma}}'' \\ \dot{u}_a \\ \dot{u}_w \end{bmatrix} = \begin{bmatrix} D_{\sigma\sigma} & D_{\sigma a} & D_{\sigma w} \\ D_{a\sigma}^* & D_{aa}^* & D_{aw}^* \\ D_{w\sigma}^* & D_{wa}^* & D_{ww}^* \end{bmatrix} \cdot \begin{bmatrix} \dot{\boldsymbol{\epsilon}} \\ \dot{m}_a \\ \dot{m}_w \end{bmatrix} \quad (3.36)$$

3.3.3 Independent Constitutive Driver

The primary objective of this driver is to verify the correctness, stability, and predictive capabilities of the constitutive formulation under controlled loading conditions, typically representative of laboratory tests. This preliminary stage allows a transparent inspection of the constitutive response, isolated from numerical issues associated with boundary value problems.

The driver is organised in a modular manner. A main execution routine controls the analysis by selecting the loading programme (e.g. monotonic or cyclic), defining the control mode, and advancing the simulation through incremental loading steps. The imposed laboratory loading conditions are enforced through a set of constraint matrices that prescribe the admissible combinations of stress, strain, and hydraulic variables, thereby enabling the simulation of stress-controlled, strain-controlled, or mixed-control tests within a unified framework.

Within this formulation, the constraint matrices $\mathbf{S}$, $\mathbf{E}$, and $\mathbf{V}$ are computed from the laboratory test definition following the framework proposed by Bardet and Choucair [9]. These matrices define the kinematic and static constraints governing the evolution of stresses, strains, and hydraulic variables during each loading increment and provide a systematic means to translate laboratory boundary conditions into a set of coupled ordinary differential equations.

At the beginning of the analysis, the complete material state is initialised, including stresses, strains, internal variables associated with plasticity and hardening, as well as variables describing the hydraulic state under unsaturated conditions. The initial values of all state variables are consistent with the imposed constraints and loading path, ensuring a well-defined initial condition and full reproducibility of the simulations.

At each loading increment, the constitutive response is obtained by directly integrating the governing system of ordinary differential equations that describe the evolution of stresses, strains, and internal variables. The formulation is expressed in rate form, and no return-mapping or projection onto the yield surface is enforced. Instead, the state variables evolve continuously according to the constitutive rates, which is consistent with explicit integration strategies commonly adopted in constitutive drivers [135].

The numerical integration of the constitutive equations is performed using an explicit Runge–Kutta–Fehlberg (RKF) scheme of order 5(6), equipped with automatic control of the local truncation error. In addition, an explicit sub-stepping strategy is employed when strong nonlinearities or abrupt changes in material response are detected. These features allow the integration to remain stable and accurate in the presence of evolving yield surfaces, activation of plastic flow, and coupled hydro-mechanical effects.

The constitutive response is evaluated through dedicated routines that compute the incremental behaviour of the material under unsaturated conditions. Depending on the value of the yield function, the response is identified as elastic or elasto-plastic, and the corresponding hypoelastic or hypoelasto-plastic relations are applied. The yield condition is defined in terms of a material-specific yield surface, whose position and shape evolve as functions of the current stress state, internal hardening variables, and hydraulic state variables. This formulation allows different constitutive models and yield surface definitions to be incorporated within the same numerical framework without modifying the underlying integration strategy.

Additional procedures are incorporated to enhance numerical robustness and physical consistency. These include the detection of loading and unloading transitions, identification of wetting and drying paths, monitoring of numerical drift with respect to the yield surface, and management of the evolution of generalised internal variables. Together, these mechanisms ensure a stable and coherent constitutive response during complex loading paths, including non-monotonic and coupled hydro-mechanical histories.

The general algorithm adopted for the explicit integration of the constitutive equations is summarised in Table 3.1. All simulations are conducted under stress-controlled conditions, where the stress vector comprises the effective stress components $\boldsymbol{\sigma}'$ and suction s , while the corresponding strain variables are the strain tensor $\boldsymbol{\varepsilon}$ and the volumetric water content

n_w . The table provides a concise overview of the algorithmic workflow described above and serves as a reference for the sequence of operations performed during each loading increment.

Table 3.1 Explicit Integration Scheme for the Numerical Implementation

Step	Description
1	Initialize variables: $y_n (\sigma'_0, s_{c0})$.
2	Compute $\mathbf{S}$, $\mathbf{E}$, and $\mathbf{V}$ based on the laboratory test definition.
3	Calculate trial state y_{tr} using $\Delta\sigma'$ and Δs .
4	Compute yield surfaces for the initial state, f_{ky_n} , and the trial state, $f_{ky_{tr}}$.
5	Check plasticity: – Identify the drying or wetting path based on Δs and update δ_{in}. – Recalculate the trial state y_{tr} with the updated hydraulic response and compute $f_{ky_{tr}}$. – Assess the response: • If $f_{ky_{tr}} \leq \text{TOL}$ (elastic), set $y_{n+1} = y_{tr}$ and proceed. • If $f_{ky_{tr}} > \text{TOL}$ (plastic), correct y_{tr} and update y_{n+1}.
6	Compute the incremental response and update y_{n+1} .
7	Update state variables: $y_n \leftarrow y_{n+1}$.

3.3.4 Verification of the UNSAT Constitutive Model

The calibration of the implemented constitutive model was performed by directly comparing the numerical results obtained in this study with the numerical simulations reported by Liu and Muraleetharan [87]. In particular, the same material parameters and loading conditions adopted by the authors were employed, and the calibration was assessed through a one-to-one comparison with their reproduced numerical responses. Undrained triaxial tests were numerically simulated on Minco silt and Toyoura sand under partially saturated and fully saturated conditions, respectively, following the experimental procedures described in Liu and Muraleetharan [88]. The material parameters provided by the authors (see Table 3.3) were adopted without further modification to ensure full consistency with the original experimental data and to allow a direct assessment of the predictive capability of the implemented formulation.

For the partially saturated cases, three undrained triaxial simulations on Minco silt (Test No. 1, Test No. 2, and Test No. 5) were performed under **constant water content (CW)**

conditions, with the pore-air pressure (p_a) kept constant throughout loading. These tests were selected to examine the influence of different initial stress and suction conditions on the coupled hydro-mechanical response of the soil. The initial conditions adopted in each simulation are summarised in Table 3.2.

Table 3.2 Initial conditions.

Test	Type	w (%)	γ_{dry} (kN/m ³)	Void ratio e	p (kPa)	Suction (kPa)
1	CW	13.81	14.49	0.814	50	30
2	CW	13.95	15.16	0.733	150	30
5	CW	11.87	14.45	0.819	50	100
1	UU	-	-	0.735	100, 1000	-
2	UU	-	-	0.735	2000, 3000	-

Initially, Test No. 1 was simulated. The results are presented in Figures 3.1, 3.2, 3.3 and 3.4. Figures 3.1 and 3.2 show that the implemented constitutive model is able to accurately reproduce the mechanical response reported by Liu and Muraleetharan [87]. In particular, an excellent agreement is obtained for both the q - p relationship and the deviatoric stress evolution with axial strain, indicating that the stress update algorithm and the elasto-plastic formulation have been correctly implemented within the constitutive driver. In contrast, discrepancies become evident when examining the volumetric and hydraulic responses. As illustrated in Figs. 3.3 and 3.4, the predicted volumetric strain and suction evolution under constant water content (CW) conditions do not exhibit the same level of agreement when the original parameter set is employed, with deviations becoming more pronounced at medium to large axial strains.

To investigate the origin of these discrepancies, a systematic parametric analysis was conducted on the unsaturated coupling parameters governing the hydro-mechanical response of the model. The analysis focused in particular on the parameter B_0 , which controls the coupling between volumetric straining and suction evolution under constant water content conditions.

As shown in Eq. 2.36, B_0 acts as the reference scaling parameter of the scalar coefficient B . This coefficient directly affects the dilatancy coefficient D , defined in Eq. 2.35, which governs the plastic volumetric strain rate through Eq. 2.34. Consequently, changes in B_0 modify the magnitude of dilatancy-induced volumetric plastic deformation and, in the unsaturated formulation, affect the associated suction evolution and degree of saturation. The parametric analysis of B_0 was therefore performed to evaluate whether the discrepancies

observed in the numerical response were related to the intensity of the hydro-mechanical coupling introduced through the dilatancy mechanism.

Table 3.3 Material parameters for Minco silt and Toyoura sand.

Parameter list		Minco silt	Toyourea sand
Elastic parameters	K_0 (kPa)	20,000	20,000
	G_0 (kPa)	30,000	10,000
	b_1	0.5	0.5
	d_1	0.5	0.5
Critical state parameters	e_{cr}	0.75	0.935
	λ	0.1	0.019
	ξ	0.7	0.7
	M_c^c	1.45	1.29
	M_e^c	1.45	1.29
Other state parameters	k_c^b	2.0	0.5
	k_c^d	1.5	3.0
	k_e^b	2.0	2.0
	k_e^d	3.0	0.07
Hardening parameters	h_0	30	500
	m	0.25	0.40
	c_m	1.0	0.0
	I_0 (kPa)	5,000	5,000
	β	20	20
	c_v	0.5	0.5
	ϖ	10	10
Dilatancy parameters	ζ	10	10
	B_0	0.1	0.5
Fabric parameters	c_f	5.0	100
	$F_{\max}$	500	100
SWCCs model parameters	b_2	8.6	2.3
	d_2	0.7	1.05
	b_3	100	4.0
	d_3	0.5	1.3
	Γ^e (kPa)	-2,500	-100
	H	5.0	7.0
	n_{ws}	0.413	0.95
	n_{wr}	0.001	0.03

The parametric study revealed that the predicted volumetric deformation and suction response are highly sensitive to the value of B_0 . When the original value proposed by Liu and Muraleetharan [87] ($B_0 = 0.1$) was adopted, the model was unable to accurately reproduce the volumetric strain and suction evolution observed in their numerical simulations, especially at medium to large axial strains. Based on this sensitivity analysis, the value of the coupling parameter was increased to $B_0 = 0.7$, which resulted in a marked improvement in the agreement with the reference results for both the ϵ_v – ϵ_a and suction–strain relationships.

It is worth noting that the modification of B_0 has a negligible effect on the deviatoric stress response, which explains the consistently good agreement observed in the stress-based comparisons (e.g. q – p and q – ϵ_a relationships). This confirms that the adjustment primarily affects the hydro-mechanical coupling mechanism, without altering the accuracy of the mechanical stress–strain response governed by the elasto-plastic formulation.

At this stage, it was not possible to unambiguously determine whether the remaining discrepancies are primarily associated with implementation-related aspects of the present constitutive driver or with ambiguities in the numerical procedures adopted in the original study. Several possible sources may contribute to these differences. First, the treatment of the volumetric component of the plastic flow may play an important role, since the plastic volumetric strain rate directly controls the suction evolution under constant water content conditions. Therefore, small differences in the integration algorithm, in the evaluation of the dilatancy coefficient, or in the update of the plastic flow direction may become evident mainly in the volumetric and hydraulic responses, while leaving the deviatoric stress response nearly unchanged.

Second, although the constitutive framework explicitly incorporates hydraulic hysteresis, uncertainties remain regarding the precise numerical realization of the hysteresis mechanism and the enforcement of the constant water content constraint. In particular, discrepancies may arise from the initialization of the hydraulic state variables within the hysteretic retention cycle. A small mismatch in the initial pair of suction and degree of saturation may place the material at a different point of the scanning curve, thereby affecting the initial hydraulic stiffness, the subsequent suction update, and the stress state at which yielding is activated.

In the present implementation, suction evolution is governed by the hardening-related term in Eq. 2.33, which links volumetric deformation to changes in degree of saturation through an effective macroscopic relationship. While this approach captures the dominant trend of suction reduction, deviations may arise depending on the specific algorithmic treatment adopted. Moreover, the use of the volumetric water content within the effective stress variable represents an additional source of numerical complexity, particularly when the

constant water content constraint has to be enforced simultaneously with the stress update, suction evolution, and hysteretic water retention response.

Moreover, the original work by Liu and Muraleetharan [87] provides limited details regarding the numerical implementation of the suction update and the enforcement of the CW constraint, making an exact one-to-one replication challenging. Consequently, part of the observed mismatch in volumetric and hydraulic responses should be attributed to unavoidable interpretation- and implementation-dependent choices, rather than to intrinsic limitations of the constitutive framework itself. For this reason, all subsequent simulations presented in this chapter were performed using the recalibrated value $B_0 = 0.7$.

The corresponding numerical results, including stress paths, stress–strain relationships, volumetric response, and suction evolution, are presented in Figures 3.5–3.12. Figures 3.5–3.12 show the numerical and experimental results corresponding to the CW tests on Minco silt. Overall, a very good agreement is observed between the numerical simulations and the experimental data, demonstrating the capability of the model to capture the coupled hydro-mechanical behavior of partially saturated soils.

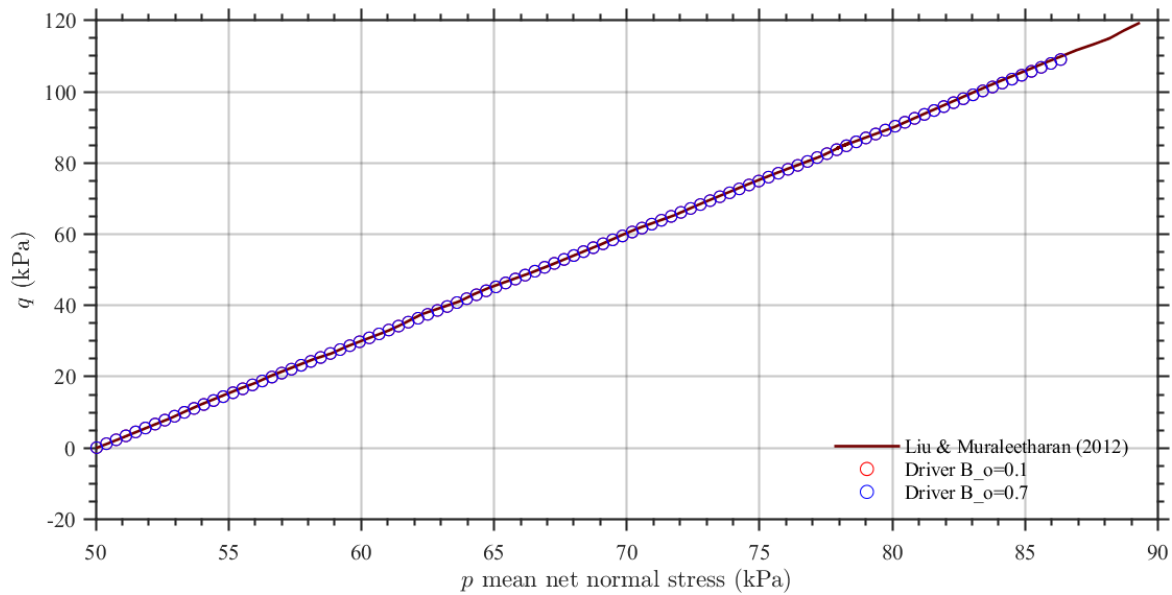


Fig. 3.1 Test No. 1 CW deviatoric stress q vs. mean net stress p .

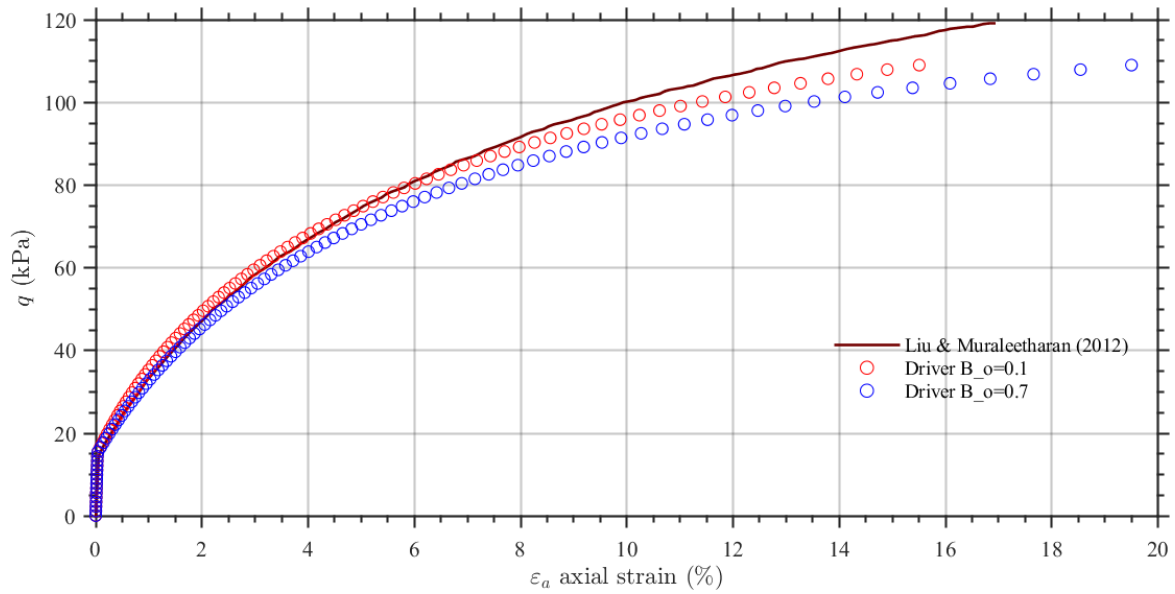


Fig. 3.2 Test No. 1 CW deviatoric stress q vs. mean net stress p .

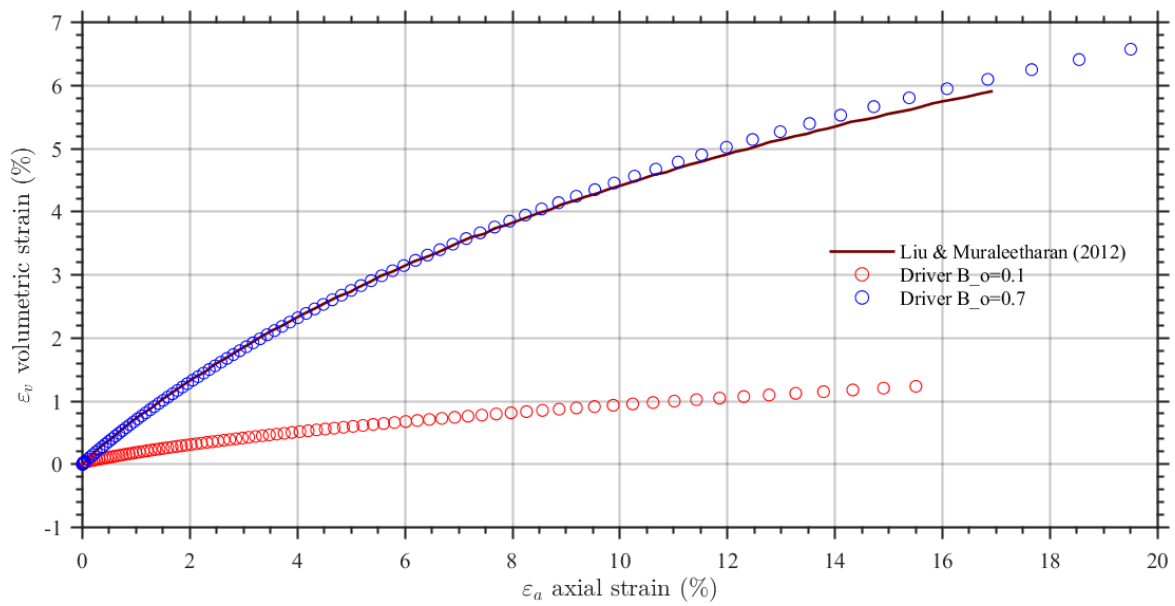


Fig. 3.3 Test No. 1 CW deviatoric stress q vs. mean net stress p .

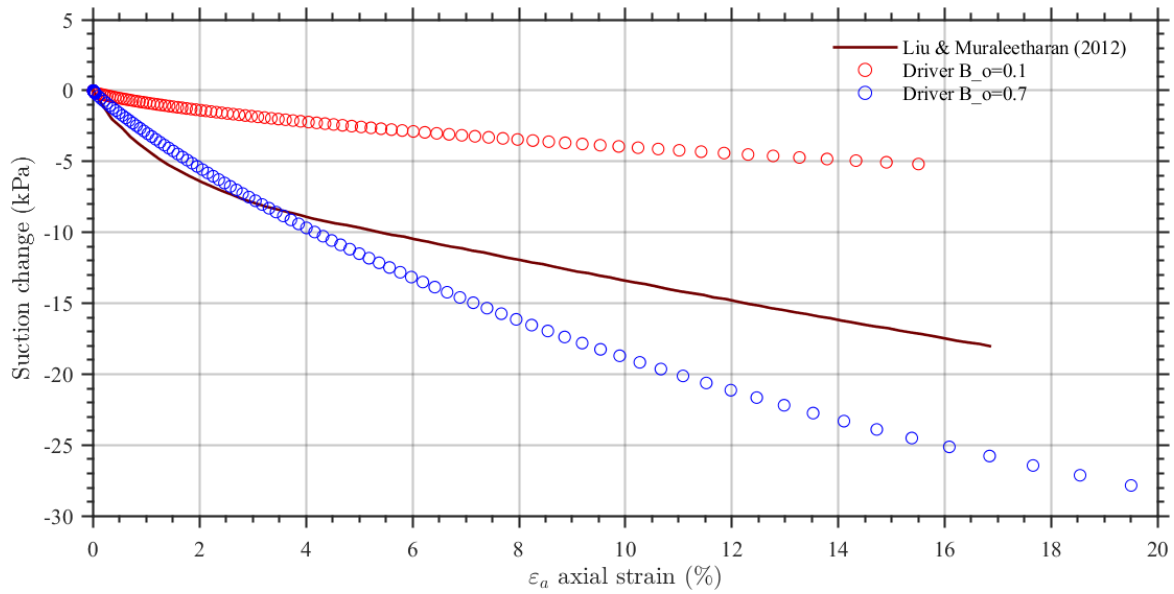


Fig. 3.4 Test No. 1 CW deviatoric stress q vs. mean net stress p .

The deviatoric stress–mean net stress (q – p) relationships exhibit nearly linear stress paths that are well reproduced by the constitutive model for all CW tests. This response reflects the dominant role of elastic stiffness and isotropic hardening during monotonic loading and indicates that the formulation accurately captures the evolution of the stress state under varying confinement conditions.

The deviatoric stress versus axial strain curves show a nonlinear increase of q with axial strain, characteristic of progressive hardening. The numerical results closely follow the numerical simulations performed by Liu and Muraleetharan [87] in terms of initial stiffness and overall stress development. Minor discrepancies are observed at large axial strains, particularly for Test No. 5, which may be attributed to the increased influence of suction-related effects that are not fully represented at the macroscopic constitutive level.

The volumetric strain response indicates a contractive behavior throughout loading for all CW tests, which is typical of fine-grained partially saturated soils. The model successfully reproduces both the magnitude of volumetric contraction and the progressive reduction of the volumetric strain rate with increasing axial deformation.

A monotonic decrease in suction is observed during loading, reflecting the coupled influence of stress redistribution and volumetric contraction. The numerical simulations capture the general trend of suction reduction with good accuracy, although a slight underestimation of the rate of suction decrease is noted at larger strains, particularly for tests with higher initial suction.

A comparison among Tests No. 1, No. 2, and No. 5 highlights the influence of the initial mean stress and initial suction on the mechanical response. Higher initial confining pressure leads to increased stiffness and higher deviatoric stress levels, while higher initial suction results in an apparent increase in strength and a more pronounced suction evolution. These effects are consistently captured by the model, confirming the internal coherence of the constitutive formulation.

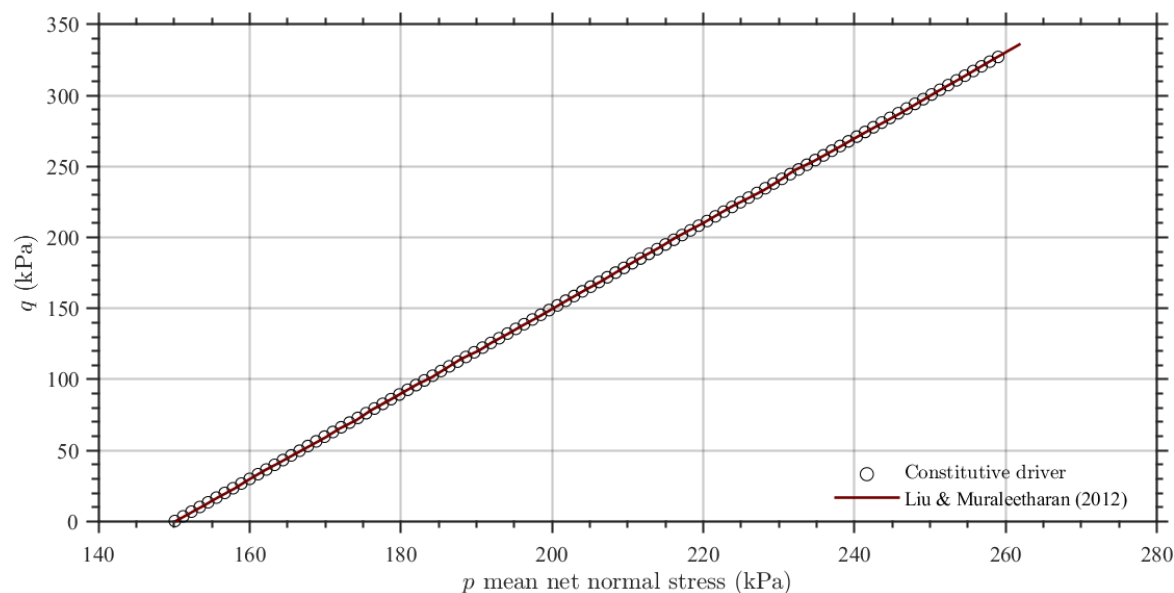


Fig. 3.5 Test No. 2 CW deviatoric stress q vs. mean neat stress p .

For the fully saturated cases, undrained triaxial simulations were performed on Toyoura sand with an initial void ratio of $e_0 = 0.735$. Confining pressures of 0.1 MPa, 1 MPa, 2 MPa, and 3 MPa were applied, in accordance with the experimental program reported by Liu and Muraleetharan [88]. These simulations were used to verify the ability of the implemented formulation to reproduce the mechanical response of saturated granular soils over a wide range of stress levels.

Figures 3.13 and 3.14 present the results of the undrained triaxial tests on Toyoura sand under fully saturated (UU) conditions, in terms of the q - p stress paths and the evolution of deviatoric stress with axial strain, respectively. Overall, the model is able to accurately reproduce the reference numerical results reported by Liu and Muraleetharan [87] for the different confining pressures. In order to achieve this level of agreement, the value of the coupling parameter B_0 was increased from 0.5 to 3.0.

As it was mentioned before for the partially saturated cases, this adjustment primarily affects the volumetric response under undrained conditions and has a negligible influence

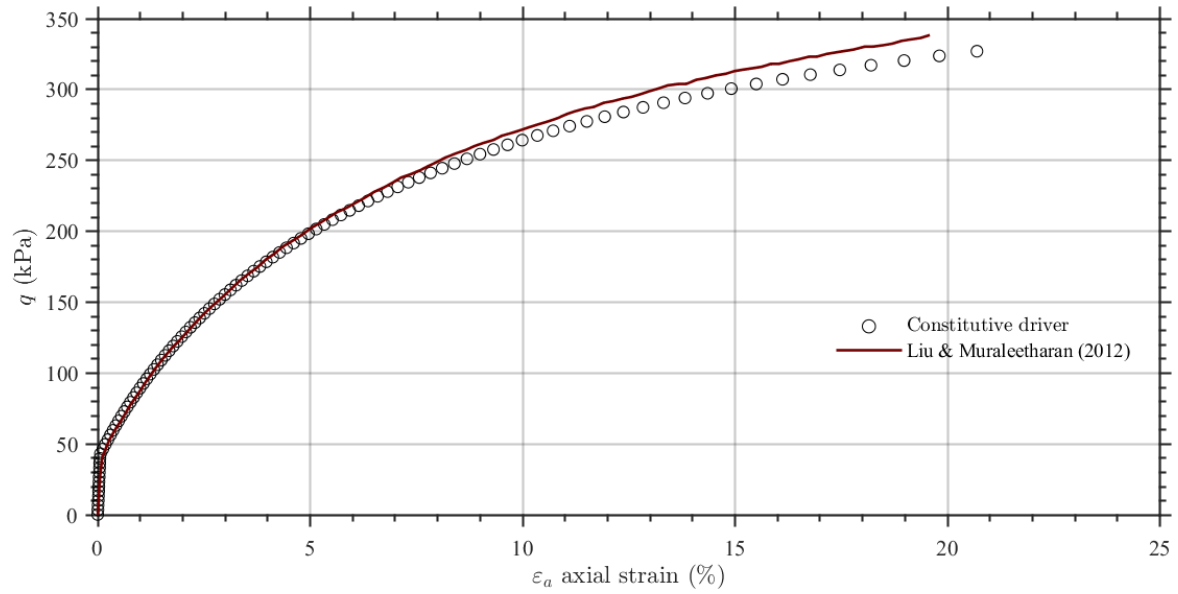


Fig. 3.6 Test No. 2 CW deviatoric stress q vs. axial strain ϵ_a .

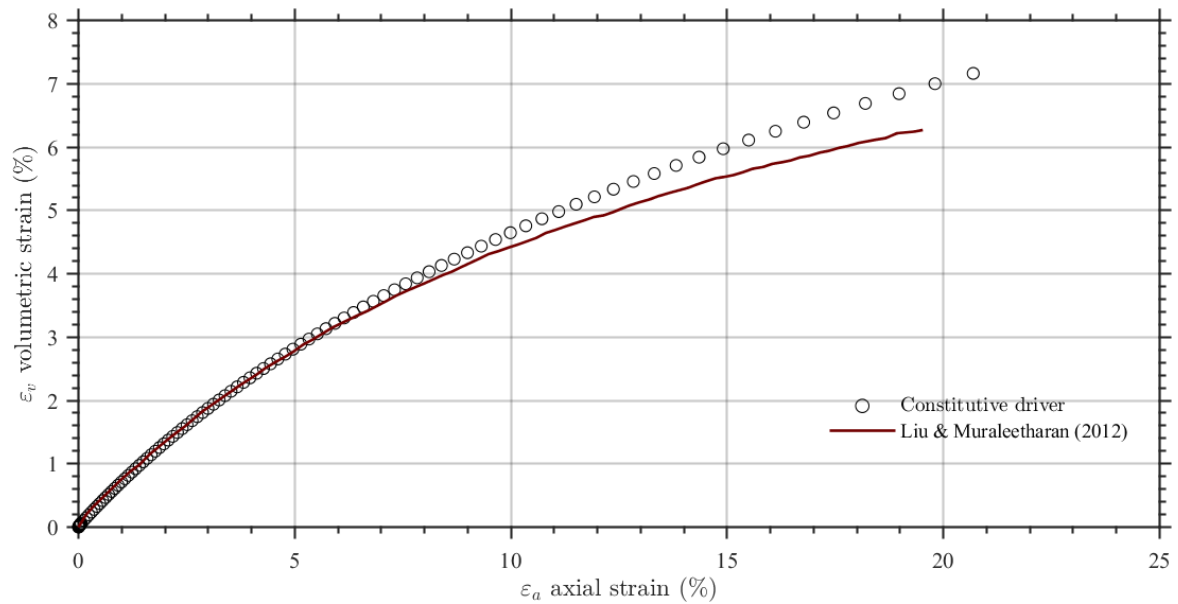


Fig. 3.7 Test No. 2 CW volumetric strain ϵ_v vs. axial strain ϵ_a .

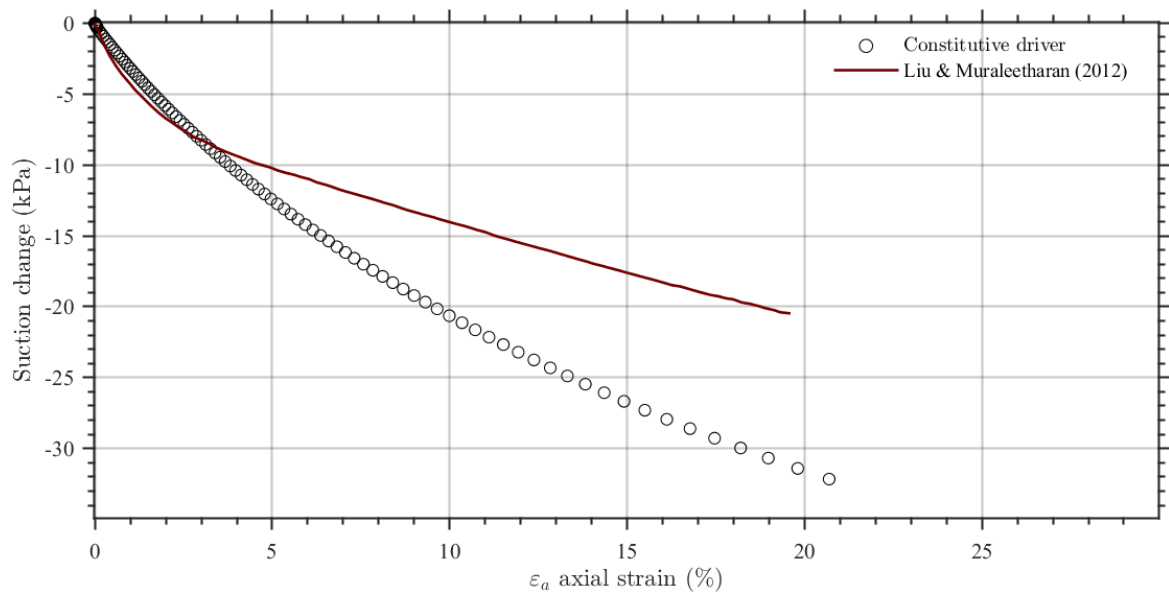


Fig. 3.8 Test No. 2 CW suction change vs. axial strain ϵ_a .

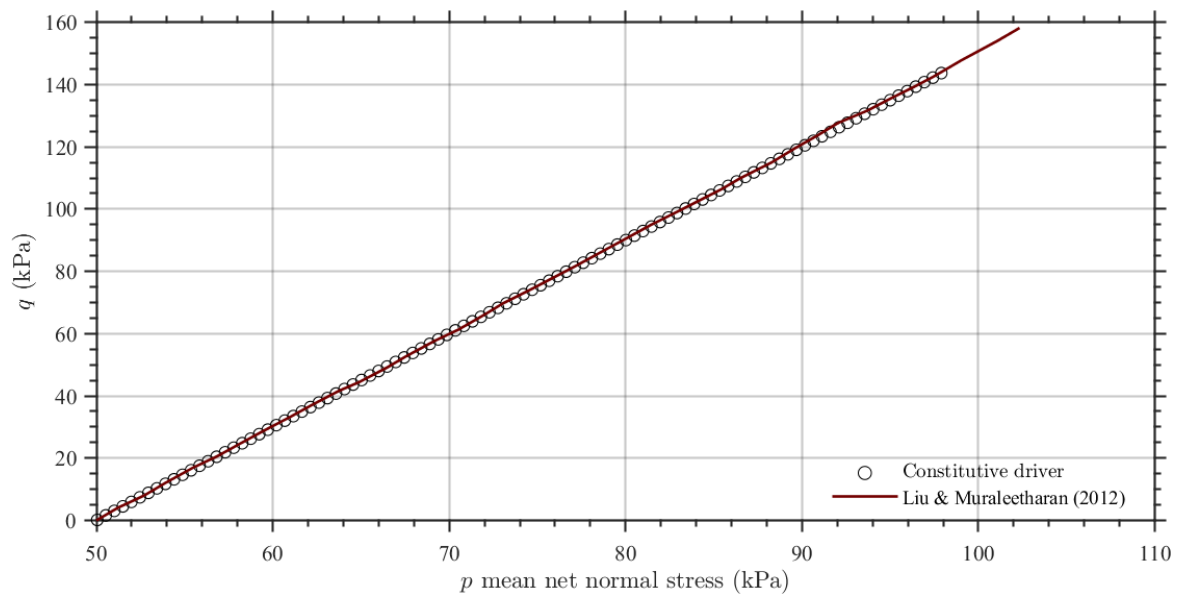


Fig. 3.9 Test No. 5 CW deviatoric stress q vs. mean neat stress p .

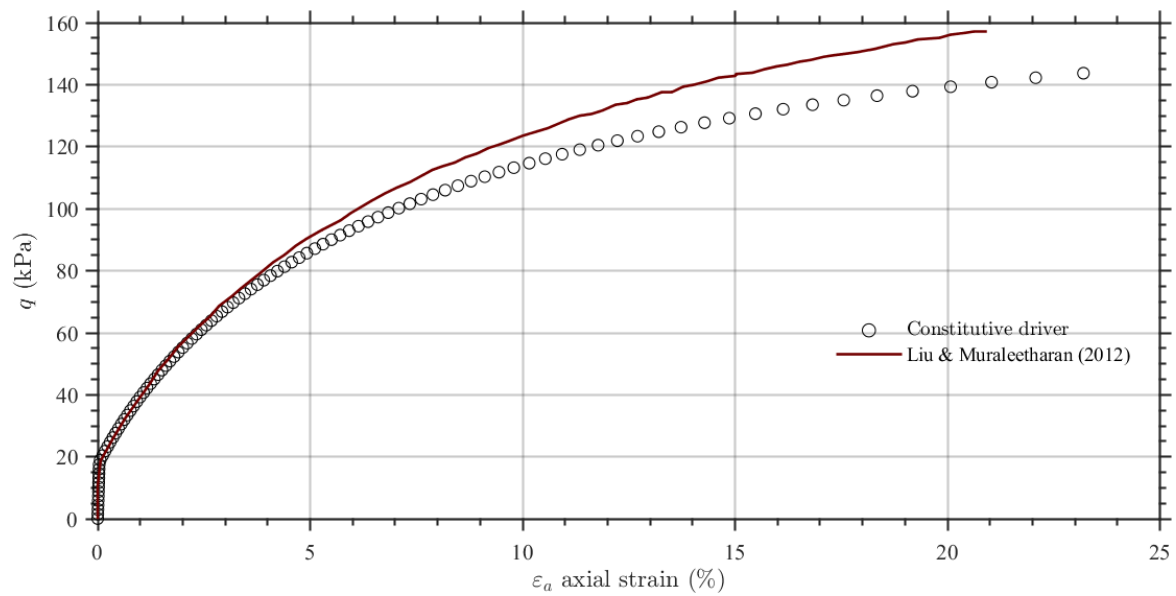


Fig. 3.10 Test No. 5 CW deviatoric stress q vs. axial strain ε_a .

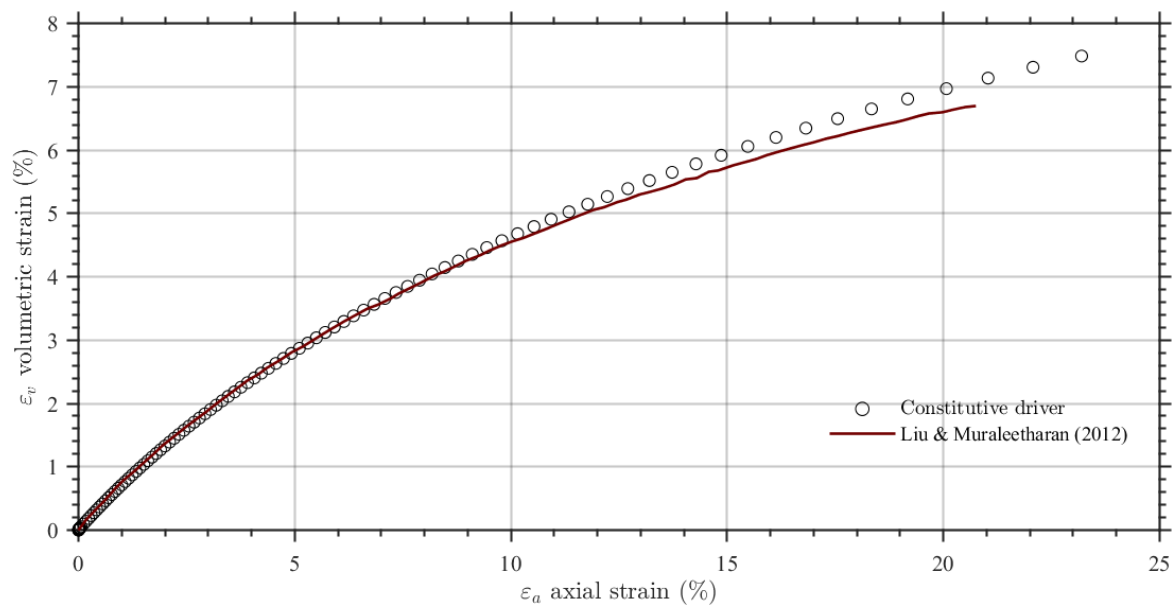


Fig. 3.11 Test No. 5 CW volumetric strain ε_v vs. axial strain ε_a .

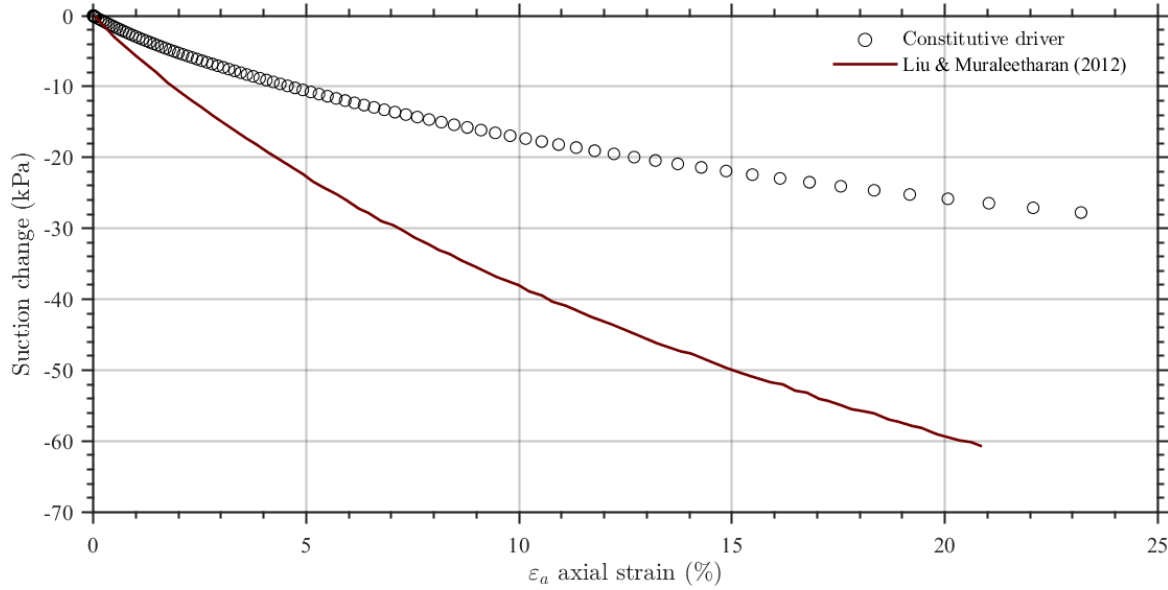


Fig. 3.12 Test No. 5 CW suction change vs. axial strain ϵ_a .

on the deviatoric stress evolution, thereby preserving the accuracy only of the mechanical response while ensuring consistency with the reference simulations.

More subtle differences can be identified during the transition from elastic to elasto-plastic behaviour, which is most clearly observed at small axial strains. In Fig. 3.14, this transition is reflected in a slight discrepancy in the initial curvature of the q - ϵ_a response, where the implemented model exhibits a more gradual activation of plastic deformation compared to the reference results. This behaviour suggests a gradual transition into plasticity, which may be associated with the numerical treatment of the elastic-plastic boundary within the adopted integration scheme.

Consistent observations can be made from Fig. 3.13, where the initial stress path in the q - p plane shows small differences in the location and sharpness of the departure from the elastic regime. In particular, the change in slope marking the onset of plasticity occurs at slightly different stress levels, depending on the integration strategy. These differences are primarily attributed to the continuous evaluation of the yield condition during explicit integration, rather than to an exact projection of the stress state onto the yield surface, as would be the case in classical return-mapping algorithms.

In summary, the verification exercises indicate that the implemented formulation is able to reproduce, on average, the main features of the coupled hydro-mechanical response reported by Liu and Muraleetharan [87], particularly in terms of stress paths and deviatoric stress evolution. However, the sensitivity of the volumetric and hydraulic responses to specific coupling parameters, together with residual ambiguities in the numerical treatment

of the constant water content constraint, motivated a critical reassessment of the complete constitutive framework.

In the present implementation, although a cap-type closure was initially introduced to ensure mathematical closure of the yield surface, it was observed that activation of this mechanism occurs only under very specific and highly idealized loading paths, similar to those discussed by Taiebat and Dafalias [136]. These conditions are not typically encountered in practical boundary value problems of engineering relevance, particularly within the range of stress levels considered in this study. Moreover, the inclusion of the cap introduces additional numerical complexity and parameter sensitivity without providing a clear improvement in the predictive capability for the stress paths and loading conditions of primary interest.

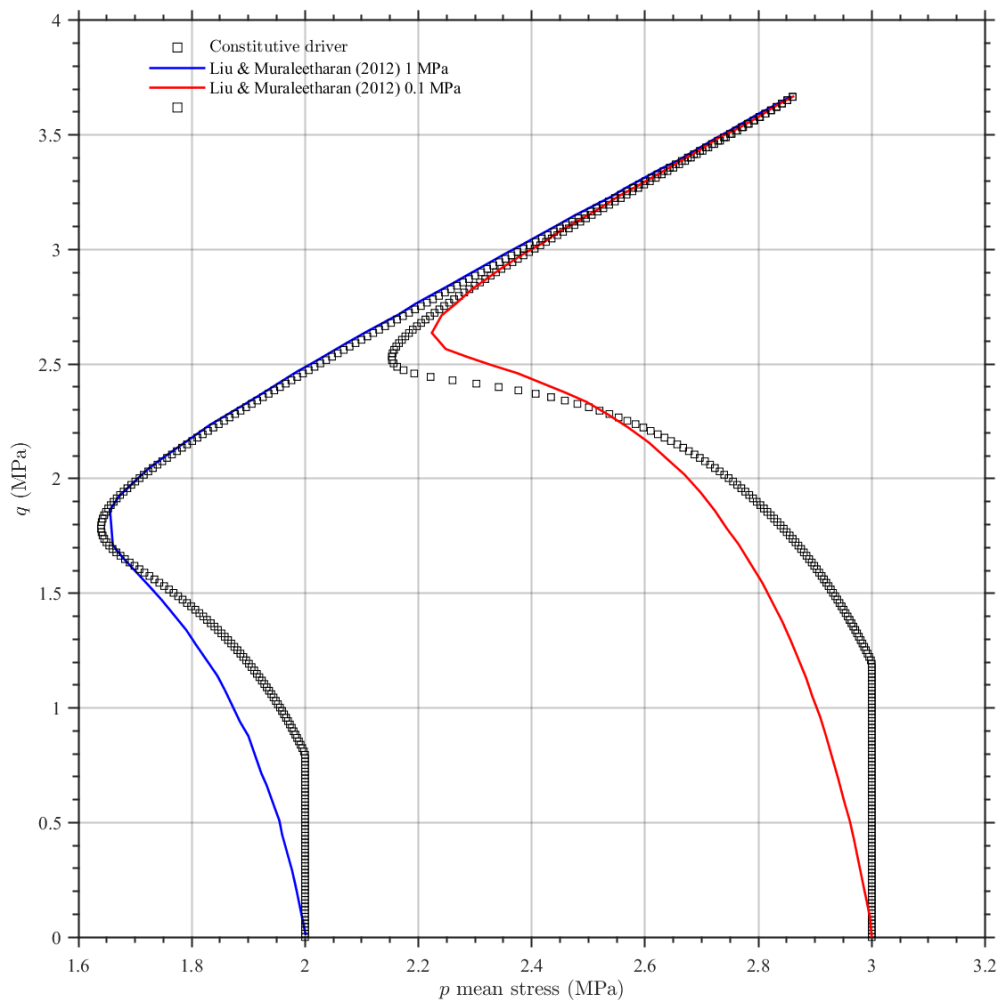


Fig. 3.13 Test No. 2 UU: deviatoric stress q vs. mean neat stress p .

For these reasons, and in line with the original SANISAND philosophy where plastic deformation is primarily driven by stress-ratio changes, the subsequent OpenSees imple-

mentation adopts a simplified formulation based on the stress-ratio-controlled yield surface originally proposed by Dafalias and Manzari [30]. The essential physical mechanisms governing anisotropy, hardening, and dilatancy are preserved, while unnecessary numerical complications associated with the cap-type mechanism are avoided. Further details of this simplified formulation are presented in section 3.4.3.

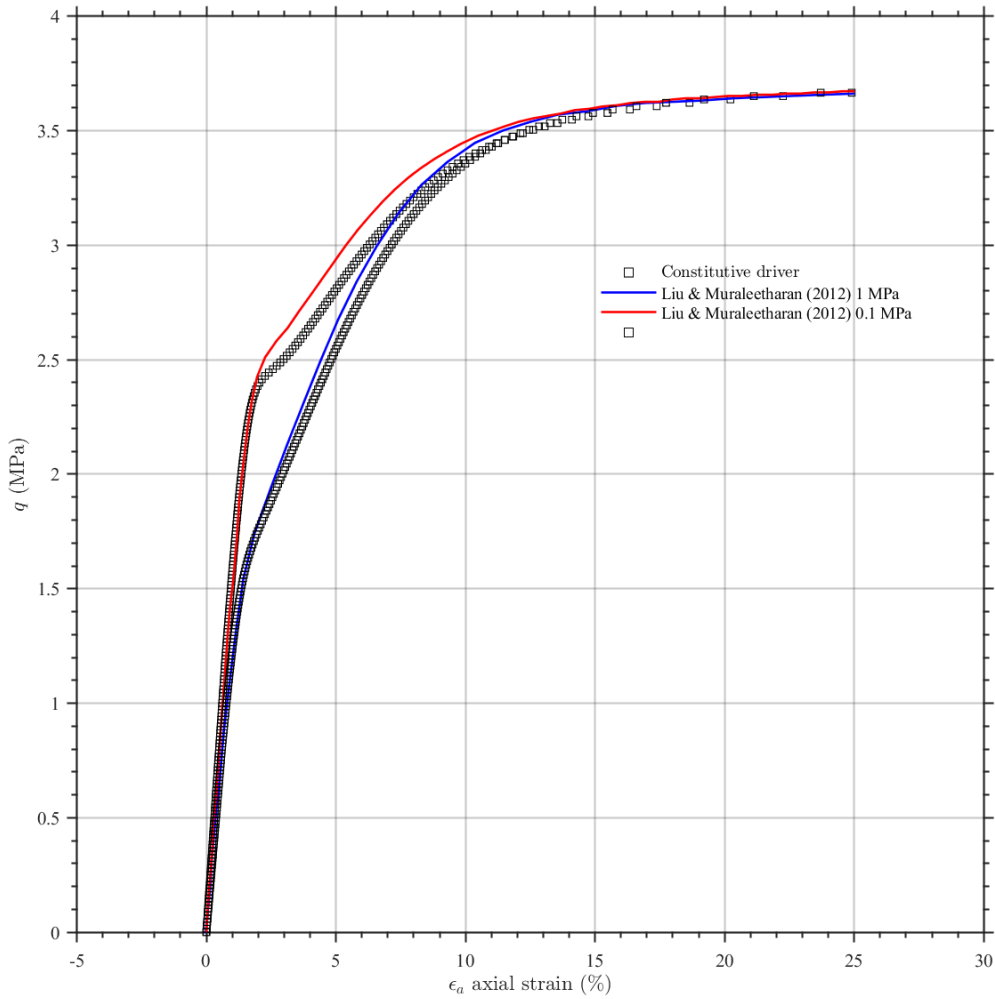


Fig. 3.14 Test No. 2 UU: deviatoric stress q vs. axial strain ϵ_a .

3.4 Implementation in OpenSees

In Geotechnical Earthquake Engineering, numerous numerical codes based on Finite Element (FE) and Finite Difference (FD) methods have been developed, with increasing reliability in the simulation of liquefaction through fully coupled nonlinear effective stress analyses.

In this study, the open-source FE platform OpenSees was selected due to its flexibility, advanced simulation capabilities, and the accessibility of its source code, which facilitates the development and extension of constitutive models. OpenSees has been successfully employed in numerous previous studies to reproduce liquefaction-related phenomena using a variety of constitutive formulations.

Constitutive models are ultimately intended for application in numerical analyses of real engineering problems; therefore, their reliability depends on their ability to reproduce a wide range of observed soil behaviours and to be implemented efficiently within numerical platforms. Model validation should thus be regarded as an iterative and ongoing process.

Based on these considerations, although several advanced sand constitutive models are already available in OpenSees, none of the existing implementations fully accounts for the hydro–mechanical coupling required to simulate the behaviour of partially saturated soils under dynamic loading.

For this reason, the constitutive model proposed by Liu and Muraleetharan [87] was implemented within the OpenSees framework, introducing minor modifications to its original formulation. The resulting model is herein referred to as `ManzariDafaliasIsoHard`.

These modifications were introduced to simplify the hydraulic description of the original formulation. It was observed that the simplified hydraulic treatment and the original fully coupled formulation reproduce an equivalent macroscopic response for the class of boundary value problems considered in this study.

In the following sections, the OpenSees environment is first introduced, followed by an outline of the general procedure for implementing constitutive models within the framework. The proposed constitutive model is then described in detail, together with the commands required for its use in OpenSees.

3.4.1 The OpenSees Environment

All numerical simulations performed in the next sections were carried out using the Open System for Earthquake Engineering Simulation, commonly referred to as OpenSees [102, 101]. OpenSees is an object-oriented, open-source software framework developed to simulate the response of structural and geotechnical systems subjected to a wide range of hazards using the Finite Element method [100].

The framework is primarily written in C++, while several numerical libraries written in Fortran and C are used for material models, linear equation solvers, and finite element. OpenSees was developed by Frank McKenna and Gregory L. Fenves with funding from the Pacific Earthquake Engineering Research (PEER) Center at the University of California, Berkeley. A comprehensive reference manual is provided by Mazzoni et al. [96].

OpenSees is primarily interpreter-based and is designed to be highly modular, allowing users to extend and integrate new functionalities. The framework exploits the object-oriented programming paradigm, whereby different objects communicate to provide the desired computational functionality. These objects are hierarchically organized and represent instances of independent classes, whose behaviour is defined within the corresponding class definitions.

Input files are written using the Tcl scripting language, or alternatively Python, both of which provide a concise and flexible syntax. Each command corresponds to a procedure implemented in the OpenSees source code, allowing different components to be combined according to the requirements of the problem under investigation.

Figure 3.15 presents the general architecture of an OpenSees input file together with the associated solution workflow. The modelling stage consists of defining the structural or geotechnical domain through the specification of nodes, elements, boundary conditions, and applied loads. Subsequently, the analysis procedure is configured by selecting the numerical components that govern the solution process, namely the constraint handler, integrator, solution algorithm, system of equations solver, and convergence test. The simulation is carried out using the `analyze` command, while output and recorder objects enable the extraction and monitoring of relevant response quantities throughout the analysis.

An additional strength of OpenSees is its capability to support serial and parallel computing. Two extensions of the basic interpreter are available for this purpose: `OpenSeesSP` and `OpenSeesMP`. The former is mainly used for large-scale simulations, whereas the latter is often adopted for parametric studies. Nevertheless, the most relevant feature of OpenSees for the present study is the possibility of freely adding new materials without modifying the core structure of the framework.

3.4.2 Implementation of a New Material in OpenSees

OpenSees is structured such that the implementation of constitutive models is independent of the finite element formulation. This design allows new materials to be introduced without altering existing element codes, and vice versa. A key role in enabling this feature is played by the C++ programming language, which supports data encapsulation and run-time binding.

Three main material abstractions are available in OpenSees:

- `UniaxialMaterial`
- `SectionForceDeformation`
- `NDMaterial`

The latter is adopted in this work, as it provides the stress–strain response of a material point under multidimensional stress states. Consequently, the constitutive equations of the model must be formulated in tensorial form.

The implementation of a new multiaxial constitutive model requires at least two files: a header file (.h), which contains the declaration of methods and member variables, and a source file (.cpp), which includes the implementation of the constitutive equations. For plane strain and three-dimensional problems, additional subclasses may be defined, although this is not mandatory.

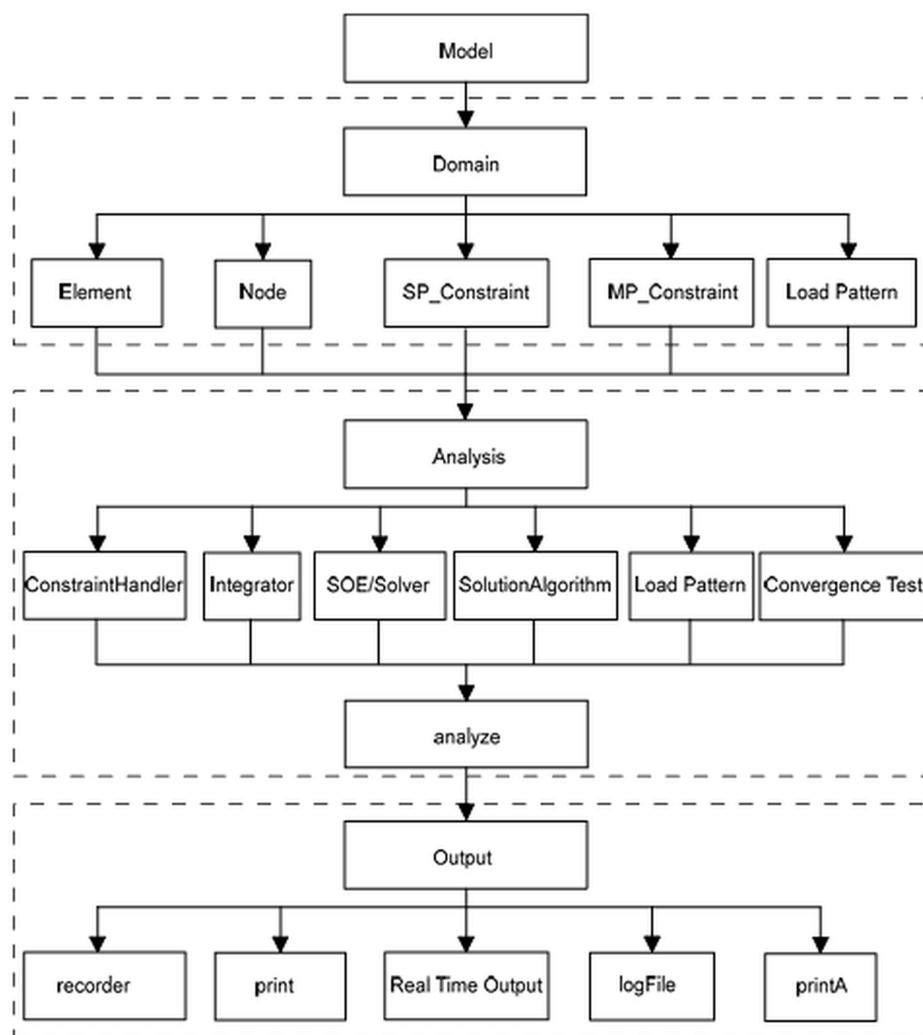


Fig. 3.15 OpenSees input file structure (adapted from McKenna et al.,[101])

The header file typically declares all public variables accessible to the framework, as well as the required standard libraries and OpenSees-specific headers. The source file defines the constructor, destructor, and all constitutive methods. Each material object is associated with

a unique integer identifier, referred to as a *tag*, which allows it to be referenced within the OpenSees domain.

The `getCopy()` method must be implemented to allow cloning of the material object. State variable management is handled through the `commitState()`, `revertToLastCommit()`, and `revertToStart()` methods. Stress, strain, and tangent stiffness tensors are communicated to the finite element through the `getStress()`, `getStrain()`, and `getTangent()` methods, respectively, all expressed in Voigt notation. Furthermore, user-defined methods can be introduced to exchange additional state variables between the element and the constitutive model. In addition to the standard interface methods, OpenSees allows user-defined functions to be introduced in order to exchange problem-specific variables between the finite element and the constitutive model. In the present work, this capability was exploited to communicate the hydraulic state variable, suction, from the element to the material. This was achieved through dedicated accessor methods (e.g. `setTrialSuction()` and `getSuction()`), enabling a consistent coupling between the element-level hydraulic formulation and the material-level constitutive response.

Finally, additional methods such as `setParameter()` and `updateParameter()` are available to support sensitivity and reliability analyses. Once implemented, the new material must be registered in the `TclModelBuilderNDMaterial` class to enable parsing from the Tcl input file.

3.4.3 Implementation of the *ManzariDafaliasIsoHard* Class in OpenSees

Based on the constitutive framework proposed by Liu and Muraleetharan [87], a simplified hydro-mechanical formulation was implemented in this work, herein referred to as the *ManzariDafaliasIsoHard* model. The objective of this implementation is to retain the essential features governing the mechanical response of granular materials under unsaturated conditions, while adopting pragmatic assumptions that enhance numerical robustness and facilitate its incorporation within a finite element framework for boundary value problems.

From a hydraulic standpoint, the soil-water retention behaviour is described using the constitutive relationship proposed by Van Genuchten [143]. The adoption of a non-hysteretic water retention model was primarily motivated by the need to simplify the overall formulation while retaining a continuous and consistent relationship between suction and degree of saturation. Moreover, field observations reported in the literature indicate that hysteretic effects may be limited under certain in-situ hydraulic regimes. For instance, Li et al. [84] observed negligible hysteresis in field-measured soil-water characteristic curves, while more recent long-term monitoring studies suggest that in-situ suction–water content paths often evolve along scanning curves and may be approximated by a single-valued relationship when

wetting–drying cycles remain within a limited suction range [67]. Consequently, the complex bounding-surface description in the suction–water content space originally introduced by Liu and Muraleetharan [87] is not explicitly reproduced. Instead, the degree of saturation and suction evolution are governed by the Van Genuchten law, which provides the hydraulic state variable required by the coupled formulation.

Regarding the hardening mechanisms, particular attention is given to the isotropic hardening parameter m , which controls the size of the yield surface in the deviatoric stress space. In the original formulation, the evolution of m includes two distinct contributions: a mechanical component associated with plastic volumetric strain and a hydraulic component linked to the irrecoverable volumetric water content, as expressed in Equation 2.28. In the present implementation, the hydraulic contribution to isotropic hardening is neglected. This assumption is justified by the fact that the Van Genuchten-based hydraulic model does not explicitly track the plastic component of the water content. As a result, the plastic water volumetric strain $\dot{n}_w^p$ is assumed to be zero, and the evolution law for the isotropic hardening parameter reduces to a purely mechanical form:

$$\dot{m} = c_m(1 + e_0)\dot{\epsilon}_v^p \quad (3.37)$$

This simplification implies that the expansion or contraction of the yield surface is governed exclusively by the accumulated plastic volumetric strain, in line with classical critical-state-based formulations. The mechanical hardening behaviour therefore remains fully consistent with the bounding surface plasticity framework, while avoiding additional hydraulic internal variables that could compromise numerical stability.

Conversely, the influence of suction is retained through its contribution to the average skeleton stress tensor. Following the conceptual interpretation proposed by Bishop, suction acts as an additional isotropic stress component that modifies the effective stress state experienced by the soil skeleton. In this model, suction does not directly affect the hardening laws nor the kinematic evolution of the yield surface. Instead, its role is confined to the definition of the average skeleton stress, which serves as the work-conjugate stress measure governing yielding, plastic flow, and elastic response. Consequently, all stress-dependent quantities—such as the yield function, plastic potential, and bounding surface definition—are evaluated in terms of the average skeleton stress. In the present simplified formulation, the yield function adopts the classical stress–ratio form of the SANISAND family, evaluated in terms of the average skeleton stress tensor. It is defined as Eq. 3.38.

$$f = \|\mathbf{s} - p\boldsymbol{\alpha}\| - \sqrt{\frac{2}{3}} m p \quad (3.38)$$

Where $\mathbf{s}$ denotes the deviatoric part of the average skeleton stress, p is the mean intergranular pressure, $\boldsymbol{\alpha}$ is the back-stress ratio tensor, and m is the isotropic hardening parameter controlling the size of the yield surface. Apart from the previously discussed simplifications in the hardening law, the remaining constitutive ingredients—plastic potential, back-stress evolution, dilatancy law, and bounding surface structure—remain consistent with the original Manzari and Dafalias [30] formulation.

This modelling choice ensures that suction influences the material response by altering the stress level at which yielding occurs, without introducing additional coupling terms in the hardening rules. Such an approach preserves the physical interpretation of suction as a stress-like variable while maintaining a clear separation between hydraulic state variables and mechanical internal variables.

In OpenSees, there are two constitutive models of the SANISAND family for liquefaction problems already available in the OpenSees framework, namely SANISAND [30] and PM4SAND [17]. Being ManzariDafaliasIsoHard model part of the SANISAND family of constitutive models, the adoption of the same structural framework used in the implementation of PM4SAND and SANISAND is natural. Accordingly, the code structure developed by Liu and Muraleetharan [87] was inherited. For this reason, the open-source nature of OpenSees was crucial.

In the ManzariDafaliasIsoHard implementation, constitutive parameters are provided through the class constructor, while all internal variables describing the material state are initialized in the `initialize()` method. The constructor defines the set of input constants governing elasticity and plasticity, namely G_0 (reference shear modulus), ν (Poisson ratio), e_{init} (initial void ratio), M_c (critical state stress ratio), c (Lode-angle shape parameter), λ_c (critical state compressibility parameter), e_0 (reference void ratio), ξ (state parameter coefficient), P_{atm} (atmospheric pressure), m_{init} (initial isotropic hardening size), h_0 (hardening modulus), c_h (hardening coefficient), n_b (bounding surface exponent), A_0 (dilatancy coefficient), n_d (dilatancy exponent), z_{max} (maximum dilatancy), c_z (dilatancy parameter), c_m (memory parameter), and ρ (mass density, `mDen` in the code). Additional numerical controls are also passed through the constructor, including the integration scheme identifier, the tangent operator type, the Jacobian type, and the tolerances Tol_F and Tol_R used in the stress update.

All tensors and internal variables required for the constitutive update are initialized in `initialize()`, including the stress and strain tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$, the elastic strain $\boldsymbol{\epsilon}^e$, the back-stress ratio tensor $\boldsymbol{\alpha}$, the fabric tensor $\mathbf{F}$, the isotropic hardening variable m_{iso} , the plastic multiplier increment $\Delta\gamma$, as well as the state-dependent moduli K and G and the tangent operators $\mathbf{C}^e$ and $\mathbf{C}^{ep}$.

Following a minimal unsaturated extension, the degree of saturation S_r and suction s are assumed to be already managed at the `NDMaterial` level and provided by the finite element. The material class was extended only to store and retrieve the Van Genuchten retention parameters supplied by the element, namely α_{VG} , m_{VG} , and the residual saturation $S_{r,res}$, with n_{VG} obtained from m_{VG} via Eq. 2.5. These quantities enable the definition of suction-dependent stress measures (e.g. average skeleton stress through Bishop-type weighting) without transferring the soil–water retention computations from the element to the material.

The tensorial quantities initialized include stress, strain, fabric, and back–stress tensors. Once the analysis starts, the trial strain is obtained from the finite element, considering either bi– or three–dimensional formulations depending on the problem type. The instructions of the `setTrialStrain()` method are contained in two additional `.cpp` files for plane strain and three–dimensional formulations of the constitutive model. Accordingly, the subclasses `ManzariDafaliasIsoHardPlaneStrain` and `ManzariDafaliasIsoHard3D` are derived.

Once the trial strain is obtained, its values expressed in Voigt notation are used within the `integrate()` method. A logical switch is implemented such that the integration can be performed either using an elastic formulation (`elastic_integrator()`) or an elastoplastic formulation based on an explicit tangent stiffness matrix (`explicit_integrator()`).

Furthermore, through the `updateMaterialStage()` command, which is common to elastoplastic constitutive models for sands, the transition from elastic to elastoplastic integration can be specified directly within the Tcl input file.

3.4.4 TCL COMMANDS

The `ManzariDafaliasIsoHard` material, whose implementation has been described in this chapter, can be invoked through the OpenSees Tcl interpreter using the following command:

```

1 nDMaterial ManzariDafaliasIsoHard $matTag \
2   $G0 $nu $e_init $Mc $c $lambda_c $e0 $ksi $P_atm \
3   $mInit $h0 $ch $nb $A0 $nd $z_max $cz $cm $rho \
4   <$integrationScheme $tangentType $JacoType $TolF $TolR>

```

Here, `$matTag` denotes the material tag. The parameters enclosed within angle brackets `<>` are optional and correspond to numerical controls of the constitutive integration. If not specified, default values are adopted. The input parameters define the elastic, plastic and numerical characteristics of the constitutive model. In particular, G_0 and ν control the elastic response, while M_c , c , λ_c , e_0 , ξ , m_{init} , h_0 , c_h , n_b , A_0 , n_d , z_{max} , c_z and c_m govern the critical state, hardening, bounding surface and dilatancy mechanisms of the model. The parameter ρ defines the mass density required for dynamic analyses. Different stress integration schemes

can be selected through the `integrationScheme` parameter. The currently available options are:

- 1: Modified Euler integration scheme with substepping.
- 4: Runge-Kutta-Fehlberg.
- 5: Forward Euler integration scheme.

The Modified Euler integration scheme with substepping is adopted as default. Additional numerical options include the selection of the tangent operator (`tangentType`), the Jacobian evaluation strategy (`JacoType`), and the tolerances `TolF` and `TolR` used to control the convergence of the stress update. The constitutive response of the material can be switched between elastic and elastoplastic formulations using the `updateMaterialStage` command, allowing a convenient separation between gravity loading and subsequent nonlinear analyses. For elastic formulation:

```
1 updateMaterialStage -material $matTag -stage 0
```

For elastoplastic formulation:

```
1 updateMaterialStage -material $matTag -stage 1
```

The dynamic phase of the analysis can be activated by switching the consolidation stage variable, as follows:

```
1 updateMaterials -material $matTag ConsolidationStage 1
```

It is noted that the degree of saturation S_r , suction s , and the Van Genuchten parameters are not provided directly through the Tcl interface of the material. These quantities are computed at the finite element level and passed to the constitutive model through dedicated interface methods, ensuring a consistent hydro–mechanical coupling while preserving the modular structure of OpenSees. In the chapters, some boundary value problems solved using the `ManzariDafaliasIsoHard` implementation are presented to assess the performance and capabilities of the proposed model.

Chapter 4

A Finite Element for Partially Saturated Soils: Formulation and Implementation

4.1 Introduction

The implementation of an advanced constitutive model within OpenSees is not sufficient, by itself, to simulate the behaviour of partially saturated soils in boundary value problems. In the OpenSees framework, the material object, i.e., each implemented constitutive model, is responsible for the stress integration at the Gauss-point level only. However, the kinematic description, the interpolation of primary variables, and the hydro-mechanical coupling at the global level are governed by the finite element formulation.

Consequently, without a properly formulated coupled finite element, a constitutive model for partially saturated soils cannot be consistently employed in practical simulations. At the time of this work, the OpenSees library did not include any element capable of handling the fully coupled solid–fluid formulation required for partially saturated media. This limitation represented a fundamental gap in the platform.

For this reason, to fill this gap, the development of a dedicated coupled finite element was a necessary and central component of this research. The objective was not merely to complement the constitutive implementation, but to extend the capabilities of OpenSees toward the analysis of boundary value problems involving partially saturated soils.

This chapter first presents the theoretical framework of mechanics of porous media, based on the formulation proposed by Zienkiewicz [175] [172], which constitutes the foundation of the coupled formulation adopted herein. Subsequently, the derivation and implementation of the finite element developed in this work, referred to as QuadUPSVG, are described in

detail. The finite element represents an evolution of the finite element formulation reported in Zienkiewicz et al. [178] that balances sake of simplicity and rigour in the analyses.

4.2 Poromechanics and Governing Equations for Partially Saturated Media

This section introduces the theoretical framework adopted to describe the hydro-mechanical behavior of partially saturated soils. The formulation is grounded in the classical poromechanics approach and provides the basis for the finite element formulation developed in the subsequent sections of this chapter.

4.2.1 Conceptual Framework for Partially Saturated Media

Soils, similarly to other granular or porous solids, can be regarded as geomaterials and analyzed through either micro-mechanical or macro-mechanical frameworks. However, due to the inherent limitations associated with analyses at the micro-mechanical scale, particularly in terms of experimental characterization and computational cost, the macro-mechanical perspective is generally preferred in geotechnical engineering applications. The latter approach allows to describe soil behaviour at the macro-scale, by considering the porous medium as a multi-phase continuum. This approach is supported by extensive experimental evidence and enables the use of well-established theoretical frameworks, such as the Theory of Porous Media (TPM) [130].

Within the macroscopic framework, the soil is modeled as a deformable porous continuum, in which the heterogeneous microstructure is replaced by an equivalent homogeneous medium. Field variables such as stress, strain, displacement, pore water pressure, and degree of saturation are defined as volume-averaged quantities over a representative elementary volume, allowing the coupled hydro-mechanical response of the material to be described at the engineering scale.

For partially saturated conditions, the pore space is occupied by two different phases: water and air. In the present work, the influence of the air phase is accounted for implicitly through saturation-dependent hydraulic properties. In particular, it is assumed that the pore air pressure remains nearly constant, an assumption commonly adopted in low-frequency geotechnical problems due to the high compressibility and mobility of air [23]. This hypothesis significantly simplifies the formulation while preserving the essential mechanisms of hydro-mechanical coupling and increases the applicability of the whole framework. However, this condition is common in real physical problems involving partially saturated soils.

The formulation developed in this chapter is restricted to the small deformation field and is applicable to static, quasi-static, and dynamic loading conditions. However, this simplification allow to model a wide range of geotechnical problems. The adopted conceptual framework provides a physically consistent basis for the derivation of the governing equations and their subsequent discretization within a finite element context.

4.2.2 Governing Equations for Porous Media

The behavior of a deformable porous medium subjected to mechanical and hydraulic loading is governed by a set of coupled partial differential equations expressing fundamental principles of conservation. Within the framework of poromechanics, these equations describe the equilibrium of the solid skeleton, the motion of the pore fluid relative to the solid phase, and the conservation of fluid mass within the pore space.

In this work, the governing equations are formulated following the classical poromechanics framework originally proposed by Zienkiewicz and Shiomi [172], which provides a unified description of static, quasi-static, and fully dynamic problems in porous media. This formulation has become the foundation of most modern finite element implementations for both saturated and unsaturated soil behavior.

The governing equations are presented in their strong form and consist of: (i) momentum balance of the porous medium, (ii) mass balance of the pore fluid, and (iii) flow conservation equation accounting for compressibility and storage effects. Under appropriate assumptions, these equations naturally reduce to the classical consolidation formulation and to purely static problems, allowing a unified theoretical and numerical framework capable of addressing a broad range of geotechnical phenomena.

The extension of the saturated formulation to partially saturated soils is achieved by incorporating additional hydraulic variables associated with the presence of multiple fluid phases in the pore space. Considering nearly constant air pressure, the formulation retains its original structure while accounting for saturation-dependent storage and flow mechanisms. The detailed expressions of the governing equations are presented in the following subsections.

4.2.2.1 Momentum Balance

The first governing equation is the total momentum equilibrium, where the overall density is expressed as $\rho = nS_w\rho_w + (1 - n)\rho_s$. A $\dot{(\cdot)}$ over a variable denotes differentiation with

respect to time. The total momentum equilibrium equation is reported in Eq. 4.1.

$$\frac{d\sigma_{ij}}{dx_j} - \rho \ddot{u}_i - \rho_f \left(\dot{w}_i + w_j \frac{dw_i}{dx_j} \right) + \rho b_i = 0. \quad (4.1)$$

In Eq. (4.1), each term has a distinct physical meaning:

- $\frac{d\sigma_{ij}}{dx_j}$ represents the stress derivatives;
- $\rho \ddot{u}_i$ denotes the inertia forces;
- ρb_i accounts for the body forces;
- $\rho_f \left(\dot{w}_i + w_j \frac{dw_i}{dx_j} \right)$ captures the relative fluid acceleration, where the second term corresponds to the convective contribution (typically small and often neglected).

In tensorial form:

$$\mathbf{S}^T \boldsymbol{\sigma} - \rho \ddot{\mathbf{u}} + \rho \mathbf{b} - \rho_f [\dot{\mathbf{w}} + \mathbf{w} \nabla^T \mathbf{w}] = 0 \quad (4.2)$$

Then, neglecting the terms associated with the last component of the expression, Eq. (4.2) reduces to:

$$\mathbf{S}^T \boldsymbol{\sigma} - \rho \ddot{\mathbf{u}} + \rho \mathbf{b} = 0 \quad (4.3)$$

4.2.2.2 Water Mass Balance

The second governing equation corresponds to the water momentum equilibrium, and it is expressed as (see Eq.4.4):

$$-\frac{dp_w}{dx_j} - R_i - \rho_f \ddot{u}_i - \rho_f \left(\dot{w}_i + w_j \frac{dw_i}{dx_j} \right) / n + \rho_f b_i = 0. \quad (4.4)$$

Here, R_i represents the viscous drag force, given by Darcy's [31] principle (see Eq. 4.5:

$$k_{ij} R_j = w_i, \quad (4.5)$$

where $k_{ij} = k$ under isotropic conditions, while in general k may depend on strain and temperature. Moreover, under partial saturation, $\rho_f = S_w \rho_w$.

In tensorial form, Eq. 4.3 can be rewritten as

$$-\nabla \mathbf{p}_w - \mathbf{R} - S_w \rho_w \ddot{\mathbf{u}} + S_w \rho_w \mathbf{b} - \rho_w (\dot{\mathbf{w}} + \mathbf{w} \nabla^T \mathbf{w}) / n = 0. \quad (4.6)$$

As it was stated earlier, the last term represents the convective fluid acceleration and is typically negligible. Neglecting this contribution, Eq. 4.6 simplifies to:

$$-\nabla \mathbf{p}_w - \mathbf{R} - S_w \rho_w \ddot{\mathbf{u}} + S_w \rho_w \mathbf{b} = 0 \quad (4.7)$$

4.2.2.3 Flow Conservation

The last equation is flow conservation for the fluid phase and is reported in Eq. 4.8, but it can be also rewritten as Eq. 4.9.

$$\frac{dw}{dx_i} + \dot{\epsilon}_{ii} + S_w \frac{n \dot{p}_w}{K_f} + S_w \frac{(1-n) \dot{p}_w}{K_s} - \frac{K_T}{K_s} (\dot{\epsilon}_{ii} + S_w \frac{\dot{p}_w}{K_s}) + n \dot{S}_w + n S_w \frac{\dot{\rho}_w}{\rho_w} + \dot{s}_0 = 0 \quad (4.8)$$

Here, different contributions are considered:

- Increased pore volume due to a change of strain assuming no change of saturation $\dot{\epsilon}_{ii}$;
- Additional volume stored by compression of pore fluid due to fluid pressure increase $S_w \frac{n \dot{p}_w}{K_f}$;
- Additional volume stored by compression of grains by the fluid pressure increase $S_w \frac{(1-n) \dot{p}_w}{K_s}$;
- Change in volume of the solid phase due to change in the intergranular effective contact stress $-d\sigma'_{ij}/K_s = -(d\sigma_{ij} + dp)/K_s = -\frac{K_T}{K_s} (d\epsilon_{ii} + S_w \frac{dp}{K_s})$, exploiting the effective stress principle and the definition of α ;
- An additional term taking into account for the change of saturation $n \dot{S}_w$.

$$\frac{dw}{dx_i} + \dot{\epsilon}_{ii} + \frac{\dot{p}_w}{Q^*} + n S_w \frac{\dot{\rho}_f}{\rho_f} + \dot{s}_0 = 0 \quad (4.9)$$

where:

$$\frac{1}{Q^*} = C_s + \frac{n S_w}{K_f} + S_w \frac{1-n}{K_s} \simeq C_s + \frac{n S_w}{K_f} \quad (4.10)$$

Here, the last term includes the bulk modulus of grains and can be neglected.

In tensorial form Eq. 4.9 can be expressed as in Eq. 4.11.

$$\nabla^T \mathbf{w} + \mathbf{I} \dot{\epsilon} + \frac{\dot{p}_w}{Q^*} + S_w n \frac{\dot{\rho}_f}{\rho_f} + \dot{s}_0 = 0 \quad (4.11)$$

The additional term C_s includes the variation in the degree of saturation is expressed in Eq. 4.12.

$$C_s = n \frac{dS_w(p_w)}{dp_w} \quad (4.12)$$

Then, the convective terms are neglected and the equation 4.11 becomes Eq. 4.13.

$$\nabla^T \mathbf{w} + \mathbf{I} \dot{\epsilon} + \frac{\dot{p}_w}{Q^*} = 0 \quad (4.13)$$

4.3 Finite Element Formulation (u–p)

The governing equations presented in the previous section define a coupled hydro-mechanical boundary value problem in terms of displacement field $\mathbf{u}$ and pore water pressure field p_w . In order to obtain a numerical solution suitable for geotechnical applications, these equations are discretized in space using the finite element method within a mixed u – p formulation.

The finite element formulation adopted in this work follows the classical approach originally proposed by Zienkiewicz and Shiomi [173] and later extended by Zienkiewicz et al. [177], which has become the basis of most u – p elements for saturated and partially saturated soils. This section briefly outlines the weak formulation of the coupled problem, the spatial discretization strategy, and the structure of the resulting discrete system, providing the conceptual framework for the specific element implementation discussed in the following subsections.

The finite element formulation is derived by recasting the governing equations into their weak form through the principle of virtual work. Appropriate test functions are introduced for both the displacement field and the pore water pressure field, and the equilibrium and flow equations are integrated over the problem domain.

The weak formulation provides the mathematical basis for the spatial discretization of the coupled problem and allows the interaction between the mechanical and hydraulic fields to be represented in a consistent and computationally efficient manner.

Following the weak formulation, the problem domain is discretized into finite elements. The displacement field $\mathbf{u}$ and the pore water pressure field p_w are interpolated independently using nodal shape functions, leading to a mixed u – p formulation.

This mixed interpolation strategy enables the coupled hydro-mechanical response of the porous medium to be captured while maintaining numerical stability and efficiency, and constitutes the standard framework for finite element modeling of fluid–solid interaction in geotechnical applications.

After spatial discretization, the coupled problem can be expressed in matrix form as a system of ordinary differential equations involving the displacement and pore pressure degrees of freedom. The resulting system includes inertia, stiffness, hydraulic coupling, and flow resistance contributions, and exhibits the characteristic block structure of u - p formulations in poromechanics.

This discrete system provides a clear physical interpretation of the interaction between the solid skeleton and the pore fluid and serves as the basis for the implementation of specific finite elements within a computational framework.

Within the u - p formulation framework, the FourNodeQuadUP element implemented in OpenSees represents a standard four-node quadrilateral element with displacement and pore pressure degrees of freedom. The formulation of this element follows directly from the poromechanics framework proposed by Zienkiewicz and Shiomi [176] and has been widely adopted for the analysis of saturated porous media.

To extend the applicability of the FourNodeQuadUP formulation to partially saturated soils, additional mechanisms associated with saturation- dependent storage and flow behavior are incorporated. Under the assumption of nearly constant pore air pressure, the original structure of the u - p formulation is preserved, while the hydraulic response is modified through appropriate constitutive relationships.

In particular, the soil–water retention behavior is described using the Van Genuchten model, allowing the degree of saturation and the hydraulic conductivity to evolve consistently with pore water pressure. These modifications lead to the development of the so-called QuadUPSVG element, which is specifically designed to capture the coupled hydro-mechanical response of partially saturated soils.

4.4 FourNodeQuadUP Formulation

In this work, the FourNodeQuadUP element is adopted as the reference formulation upon which further modifications are introduced to account for partially saturated soil behavior.

The FourNodeQuadUP element solves a coupled system of equations that links the solid skeleton displacement field with the pore pressure field. The global system takes the following form, consistent with Eqs. 3.24 and 3.28 of Chan et al. [23]. As shown in Eq. 4.14, the fully-coupled matrix form reads:

$$\underbrace{\begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix}}_{\text{Mass matrix}} \underbrace{\begin{Bmatrix} \ddot{\mathbf{u}} \\ \dot{\mathbf{p}} \end{Bmatrix}}_{\text{Damping and coupling matrix}} + \underbrace{\begin{bmatrix} \mathbf{C}_{\text{modified}} & \mathbf{Q} \\ \mathbf{Q}^T & \mathbf{H} \end{bmatrix}}_{\text{Damping and coupling matrix}} \underbrace{\begin{Bmatrix} \dot{\mathbf{u}} \\ \mathbf{p} \end{Bmatrix}}_{\text{Damping and coupling matrix}} + \underbrace{\begin{bmatrix} \mathbf{K}_T & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}}_{\text{Stiffness matrix}} \underbrace{\begin{Bmatrix} \mathbf{u} \\ - \end{Bmatrix}}_{\text{Stiffness matrix}} = \underbrace{\begin{Bmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \end{Bmatrix}}_{\mathbf{f}} \quad (4.14)$$

where:

- $\mathbf{M} \in \mathbb{R}^{8 \times 8}$: consistent mass matrix for displacements.
- $\mathbf{S} \in \mathbb{R}^{4 \times 4}$: compressibility matrix for pore pressure.
- $\mathbf{C}_{\text{modified}}$: Rayleigh-type damping contribution.
- $\mathbf{Q} \in \mathbb{R}^{8 \times 4}$: coupling matrix relating $\dot{\mathbf{u}}$ and $\mathbf{p}$.
- $\mathbf{H} \in \mathbb{R}^{4 \times 4}$: permeability matrix derived from Darcy flow.
- $\mathbf{K}_T \in \mathbb{R}^{8 \times 8}$: tangent stiffness matrix of the solid skeleton.
- $\mathbf{f}^{(1)}$: external mechanical force vector.
- $\mathbf{f}^{(2)}$: hydraulic flux boundary and source terms.

In the following, each term is described in detail.

4.4.1 Displacement Mass Matrix $\mathbf{M}$

The consistent displacement mass matrix for saturated porous media is given by Eq. 4.15. The discrete approximation of Eq. 4.15 is expressed in Eq. 4.16.

$$\mathbf{M} = \int_{\Omega} (\mathbf{N}^u)^T \rho \mathbf{N}^u d\Omega \quad (4.15)$$

$$M_{ij} \approx \sum_{m=1}^{n_{\text{gp}}} \rho^{(m)} N_i^u(x_m) N_j^u(x_m) |J(x_m)| w^{(m)} \quad (4.16)$$

where:

- $\mathbf{N}^u$: matrix of shape functions associated with nodal displacements (u_x, u_y) ,
- ρ : bulk density of the mixture (solid + fluid),
- Ω : domain of the element.

- $|J(x_m)|$: Jacobian determinant,
- N_i^u are shape functions for displacements.

4.4.2 Pore Pressure Compressibility Matrix $\mathbf{S}$

The pore pressure compressibility matrix is defined by Eq. 4.17. Its discrete approximation is given by Eq. 4.18.

$$\mathbf{S} = - \int_{\Omega} \frac{1}{K_c} (\mathbf{N}^p)^T \mathbf{N}^p d\Omega \quad (4.17)$$

$$S_{ij} \approx - \sum_{m=1}^{n_{gp}} \frac{1}{K_c(x_m)} N_i^p(x_m) N_j^p(x_m) |J(x_m)| w^{(m)} \quad (4.18)$$

where:

- $\mathbf{N}^p$: vector of shape functions associated with nodal pore pressures,
- K_c : pore-fluid compressibility (or equivalent storage bulk modulus),
- Ω : domain of the element,
- S_{ij} : (i, j) entry of the compressibility matrix $\mathbf{S}$,
- n_{gp} : number of Gauss integration points,
- $N_i^p(x_m)$: pore-pressure shape function evaluated at Gauss point x_m ,
- $|J(x_m)|$: determinant of the Jacobian matrix at Gauss point x_m ,
- $w^{(m)}$: Gauss integration weight.

4.4.3 Global Mass Matrix Structure

The assembled mass matrix adopts a block-diagonal structure as shown in Eq. 4.19.

$$\mathbf{K}_{\text{mass}} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix} \in \mathbb{R}^{12 \times 12} \quad (4.19)$$

where:

- $\mathbf{K}_{\text{mass}}$: global mass matrix of the coupled element,
- $\mathbf{M}$: displacement mass matrix associated with inertial effects of the solid–fluid mixture,

- **S**: pore-pressure compressibility (storage) matrix,
- $\mathbb{R}^{12 \times 12}$: matrix space corresponding to the total number of element degrees of freedom.

4.4.4 Mechanical Damping and Mass-Proportional Terms

The modified damping matrix is defined by Eq. 4.20.

$$\mathbf{C}_{\text{modified}} = \alpha_M \mathbf{M} + \beta_K \mathbf{K}_T + \beta_{K_0} \mathbf{K}_0 + \beta_{K_c} \mathbf{K}_c \quad (4.20)$$

where:

- $\mathbf{C}_{\text{modified}}$: effective damping matrix used in the transient analysis,
- α_M : mass-proportional Rayleigh damping coefficient,
- β_K : stiffness-proportional coefficient associated with the current tangent stiffness,
- β_{K_0} : stiffness-proportional coefficient associated with the initial stiffness matrix,
- β_{K_c} : stiffness-proportional coefficient associated with the committed stiffness matrix,
- $\mathbf{M}$: displacement mass matrix,
- $\mathbf{K}_T$: current tangent stiffness matrix,
- $\mathbf{K}_0$: initial stiffness matrix,
- $\mathbf{K}_c$: last committed stiffness matrix.

4.4.5 Coupling Matrix $\mathbf{Q}$

The coupling matrix is defined in its continuous form in Eq. 4.21. The discrete approximation of Eq. 4.21 is given in Eq. 4.22.

$$Q_{ij} = - \int_{\Omega} (\nabla N_i^u)^T N_j^p d\Omega \quad (4.21)$$

$$Q_{ij} \approx - \sum_{m=1}^{n_{\text{gp}}} |J(x_m)| w^{(m)} \left(\frac{\partial N_i^u}{\partial x} N_j^p + \frac{\partial N_i^u}{\partial y} N_j^p \right) \quad (4.22)$$

where:

- Q_{ij} : (i, j) entry of the coupling matrix $\mathbf{Q}$,

- Ω : domain of the element,
- N_i^u : displacement shape function associated with the i -th displacement degree of freedom,
- N_j^p : pore-pressure shape function associated with the j -th pressure degree of freedom,
- ∇N_i^u : spatial gradient of the displacement shape function,
- n_{gp} : number of Gauss integration points,
- $|J(x_m)|$: determinant of the Jacobian matrix at Gauss point x_m ,
- $w^{(m)}$: Gauss integration weight.

4.4.6 Permeability Matrix $\mathbf{H}$

The permeability matrix arising from Darcy's law is defined in Eq. 4.23. Its discrete form is expressed in Eq. 4.24.

$$H_{ij} = - \int_{\Omega} \nabla N_i^p \cdot \mathbf{k} \cdot \nabla N_j^p d\Omega \quad (4.23)$$

$$H_{ij} \approx - \sum_{m=1}^{n_{gp}} |J(x_m)| w^{(m)} \left(k_x \frac{\partial N_i^p}{\partial x} \frac{\partial N_j^p}{\partial x} + k_y \frac{\partial N_i^p}{\partial y} \frac{\partial N_j^p}{\partial y} \right) \quad (4.24)$$

where:

- H_{ij} : (i, j) entry of the permeability matrix $\mathbf{H}$,
- Ω : domain of the element,
- N_i^p, N_j^p : pore-pressure shape functions associated with the i -th and j -th pressure degrees of freedom,
- ∇N_i^p : spatial gradient of the pore-pressure shape function,
- $\mathbf{k}$: hydraulic conductivity tensor,
- k_x, k_y : principal components of the hydraulic conductivity in the x and y directions,
- n_{gp} : number of Gauss integration points,
- $|J(x_m)|$: determinant of the Jacobian matrix at Gauss point x_m ,
- $w^{(m)}$: Gauss integration weight.

4.4.7 Structure of the Matrix $\mathbf{K}$

The assembled matrix structure is shown in Eq. 4.25.

$$\mathbf{K} = \begin{bmatrix} \mathbf{C}_{\text{modified}} & \mathbf{Q} \\ \mathbf{Q}^T & \mathbf{H} \end{bmatrix} \in \mathbb{R}^{12 \times 12} \quad (4.25)$$

where:

- $\mathbf{K}$: global coupled matrix at the element level,
- $\mathbf{C}_{\text{modified}}$: modified mechanical damping matrix including mass- and stiffness-proportional contributions,
- $\mathbf{Q}$: hydro-mechanical coupling matrix relating displacement and pore-pressure degrees of freedom,
- $\mathbf{Q}^T$: transpose of the coupling matrix ensuring symmetry of the formulation,
- $\mathbf{H}$: permeability matrix arising from Darcy's law,
- $\mathbb{R}^{12 \times 12}$: matrix dimension corresponding to the total number of degrees of freedom (8 displacement DOFs + 4 pore-pressure DOFs).

4.4.8 Initial Stiffness Matrix

The initial stiffness matrix derived from the principle of virtual work is given in Eq. 4.26. The corresponding discrete approximation is expressed in Eq. 4.27.

$$\mathbf{K}_0 = \int_{\Omega} \mathbf{B}^T \mathbf{D}_0 \mathbf{B} d\Omega \quad (4.26)$$

$$K_{ia,ib} \approx \sum_{m=1}^{n_{\text{gp}}} |J(x_m)| w^{(m)} \mathbf{B}_{\alpha}^T \mathbf{D}_0^{(m)} \mathbf{B}_{\beta} \quad (4.27)$$

where:

- $\mathbf{K}_0$: initial (elastic) stiffness matrix at the element level,
- $\mathbf{B}$: strain–displacement matrix relating nodal displacements to strain components,
- $\mathbf{D}_0$: initial constitutive (elastic) tangent matrix,
- Ω : domain of the element,

- n_{gp} : number of Gauss integration points,
- $|J(x_m)|$: determinant of the Jacobian matrix evaluated at Gauss point x_m ,
- $w^{(m)}$: Gauss integration weight associated with point m ,
- $\mathbf{B}_\alpha, \mathbf{B}_\beta$: strain–displacement matrices associated with the corresponding displacement degrees of freedom,
- $\mathbf{D}_0^{(m)}$: elastic constitutive matrix evaluated at Gauss point m .

The tangent stiffness matrix $\mathbf{K}_T$ follows the same formulation as Eq. 4.26, replacing $\mathbf{D}_0$ by the consistent tangent constitutive matrix.

4.4.9 Extension FourNodeQuadUPSVG

The above-mentioned formulation is then extended to include the behaviour of partially saturated soils. This section documents the main enhancements and extensions implemented in the new custom finite element FourNodeQuadUPSVG, designed to account for unsaturated porous media behaviour through Van Genuchten’s formulation. The implementation is of particular interest because the finite element response is explicitly made dependent on suction through the degree of saturation and the corresponding variation of hydraulic conductivity. This represents a key difference with respect to the standard FourNodeQuadUP element already implemented in OpenSees. Although the classical QuadUP formulation may develop negative pore-water pressures, as observed in the numerical examples presented in Chapters 5 and 6, these pressures are obtained within a fully saturated framework and can only be interpreted as suction in a numerical sense. They mainly arise as a consequence of satisfying equilibrium and achieving convergence of the coupled system. Therefore, the response does not provide a fully consistent description of partially saturated behaviour.

For this reason, it is necessary to properly introduce the dependency of the global hydro–mechanical response on suction by linking pore-water pressure, degree of saturation, and permeability within the finite element formulation. These developments rely upon the theoretical framework established by Zienkiewicz et al. [174] and further discussed by Chan et al. [23], which reformulate the coupled u – p equations under partial saturation. One approach for incorporating partial saturation is to assume that the air pressure remains constant ($p_a = 0$), a simplification that Zienkiewicz identified as a robust and efficient strategy for boundary value problems. In fact, this condition frequently occurs in practical problems. By adopting this assumption, the complexity of multiphase fluid interactions in the numerical implementation is reduced, leading to a more stable and computationally efficient formulation.

For this reason, this approach was adopted in the present work, and the following sections present the specific modifications introduced to the original FourNodeQuadUP formulation.

4.4.9.1 Effective Pressure Redefinition

The effective pressure is redefined as Eq. 4.28. This approximation is valid when the air pressure is assumed negligible and saturation effects dominate the hydraulic behavior. The effective stress is still defined using Terzaghi's principle but with a modified pressure term.

$$p = \chi_w p_w + (1 - \chi_w) p_a \approx \chi_w p_w \quad \text{with } p_a = 0 \quad (4.28)$$

4.4.9.2 Modified Storage Term in Mass Conservation

In the conservation equation, additional terms appear due to changes in saturation and compressibility. The storage term Q^* is modified into the expression represented in Eq. 4.29.

$$Q^* = C_s + \frac{n S_w}{K_f} + \frac{(\alpha - n) \chi_w}{K_s} \quad (4.29)$$

where C_s is a compressibility coefficient that accounts for changes in saturation with respect to pore water pressure and it is represented in Eq. 4.30. These modifications allow for capturing both the compressibility of the fluid and the saturation-dependent storage effects.

$$C_s = n \frac{dS_w}{dp_w} \quad (4.30)$$

4.4.9.3 Soil–Water Retention and Unsaturated Permeability

In the proposed formulation, the unsaturated behavior of the porous medium is modeled by combining the soil–water retention curve (SWRC) with the relative permeability function. Since in this work the air pressure is assumed constant ($u_a = 0$), matric suction reduces to $s = -u_w$. Hence, suction is directly represented by the negative values of pore water pressure, an assumption that simplifies the formulation by avoiding the explicit treatment of the air phase. The effective saturation S_e is then expressed in Eq. 4.31. Also, it is used to compute the relative permeability according to the Van Genuchten–Mualem model (see Eq. 4.32).

$$S_e = \frac{S_r - S_r^{\text{res}}}{1 - S_r^{\text{res}}}, \quad (4.31)$$

$$k_r(S_e) = \sqrt{S_e} \left[1 - \left(1 - S_e^{1/m_{VG}} \right)^{m_{VG}} \right]^2. \quad (4.32)$$

The intrinsic permeability under unsaturated conditions is finally expressed as Eq. 4.33, where $\mathbf{k}_0$ denotes the saturated intrinsic permeability tensor.

$$\mathbf{k} = k_r(S_e) \cdot \mathbf{k}_0, \quad (4.33)$$

These constitutive relations are implemented numerically in the FourNodeQuadUPSVG element, allowing the local hydraulic conductivity at each integration point to evolve consistently with the interpolated pore water pressure. This coupling ensures a coherent representation of the interaction between the soil–water retention properties and unsaturated flow behavior.

4.5 FourNodeQuadUPSVG Implementation

The implementation of the FourNodeQuadUPSVG element was based on the existing FourNodeQuadUP formulation, already available in OpenSees. This process required revisiting the original formulation presented by Zienkiewicz [175, 174], since the FourNodeQuadUP element is grounded on that theoretical framework. For completeness, the governing formulation on which the implementation is based is summarized in Appendix A. In the following, the key aspects of the implementation of the new element are presented, highlighting the main functions required for the development of the new element. Following this formulation and building upon the work of Zienkiewicz et al. [176], later documented by Chan et al. [23], the same theoretical structure was adopted to account for the soil response under unsaturated conditions. In particular, the porous media framework proposed by Zienkiewicz et al. [176] was retained, while the soil–water retention relationship was introduced through the Van Genuchten model [143] to describe the suction—degree of saturation coupling. The corresponding modifications to the formulation and their numerical implementation are presented in the following sections.

4.5.1 New Input Parameters

The element constructor has been extended to incorporate the parameters required by the Van Genuchten model, which govern the soil–water retention behavior and the relative permeability function. These parameters are:

- α_{VG} : Inverse of the air-entry value, controlling the position of the soil–water retention curve.
- n_{VG} (or m_{VG} via Eq. 2.5): Shape parameter controlling the steepness of the retention curve.

- S_r^{res} : Residual degree of saturation, representing the lower bound of water content that cannot be removed by drainage.

4.5.2 Computation of Saturation and Relative Permeability

Internally, the degree of saturation S_r is evaluated from the interpolated pore water pressure p_w at the element centroid. This choice ensures consistency with the bilinear interpolation adopted for the pore pressure field and provides a representative hydraulic state at the element level.

Evaluating the nonlinear retention relationship at a single representative point avoids excessive intra-element variability and enhances numerical stability within the coupled formulation.

For this purpose, the following C++ functions were incorporated into the element implementation to evaluate saturation, its derivative, and the relative permeability:

Listing 4.1 Unsaturated laws with regularization

```
double FourNodeQuadUPSVG::getElementPwCenter() {
    double pnod[4] = {0,0,0,0};
    if (nd1Ptr) { const Vector& d1 = nd1Ptr->getTrialVel(); if
        (d1.Size()>=3) pnod[0]=d1(2); }
    if (nd2Ptr) { const Vector& d2 = nd2Ptr->getTrialVel(); if
        (d2.Size()>=3) pnod[1]=d2(2); }
    if (nd3Ptr) { const Vector& d3 = nd3Ptr->getTrialVel(); if
        (d3.Size()>=3) pnod[2]=d3(2); }
    if (nd4Ptr) { const Vector& d4 = nd4Ptr->getTrialVel(); if
        (d4.Size()>=3) pnod[3]=d4(2); }
    return 0.25 * (pnod[0] + pnod[1] + pnod[2] + pnod[3]);
}

double FourNodeQuadUPSVG::calculateSaturation(double pw) const {
    const double m = std::min(std::max(mVG, 1e-8), 1.0 - 1e-8);
    const double n = 1.0 / (1.0 - m);
    const double s = std::max(-pw, 0.0);           // suction in kPa
    double Se = 1.0;
    if (alphaVG > 0.0 && s > 0.0) {
        const double as_n = std::pow(alphaVG*s, n);
        Se = std::pow(1.0 + as_n, -m);
    }
}
```

```

}
const double Srmin = std::min(std::max(Sr_res, 0.0), 0.9999999);
double Sr = Srmin + (1.0 - Srmin) * Se;
return std::min(std::max(Sr, Srmin), 1.0);
}

```

The derivative of the effective degree of saturation with respect to the pore water pressure is required for the evaluation of the storage contribution in the coupled formulation. The storage coefficient is defined as Eq. 4.30. According to the Van Genuchten retention model, the derivative of the effective saturation with respect to suction s is expressed as Eq. 4.34. Since suction is defined as $s = -p_w$, the chain rule yields as Eq. 4.35.

$$\frac{dS_e}{ds} = -mn\alpha_{VG}(\alpha_{VG}s)^{n-1} [1 + (\alpha_{VG}s)^n]^{-(m+1)}. \quad (4.34)$$

$$\frac{dS_e}{dp_w} = -\frac{dS_e}{ds}. \quad (4.35)$$

To prevent numerical stiffening in the vicinity of full saturation ($s \rightarrow 0$), a regularization function is introduced in the form represented in Eq. 4.36.

$$\phi = \frac{s}{s + s_c}, \quad s_c = \frac{1}{2}\alpha_{VG}^{-1}. \quad (4.36)$$

The regularized derivative adopted in the implementation is therefore given by Eq. 4.37.

$$\left(\frac{dS_e}{dp_w} \right)_{\text{reg}} = \phi \frac{dS_e}{dp_w}. \quad (4.37)$$

Listing 4.2 Derivative dS_e/dp_w with regularization

```

double FourNodeQuadUPSVG::derivativeSe(double pw) const {
    if (alphaVG <= 0.0 || pw >= 0.0) return 0.0; // saturated
    const double m = std::min(std::max(mVG, 1e-8), 1.0 - 1e-8);
    const double n = 1.0 / (1.0 - m);
    const double s = -pw, as = alphaVG * s;
    if (as <= 0.0) return 0.0;
    const double common = std::pow(1.0 + std::pow(as, n), -(m + 1.0));
    const double dSe_ds = -m * n * alphaVG * std::pow(as, n - 1.0) *
        common;
    const double dSe_dpw = -dSe_ds; // > 0
    const double sa = 1.0 / std::max(alphaVG, 1e-12), sc = 0.5 * sa;

```

```

const double phi = s / (s + sc); // [0..1]
return phi * dSe_dpw;
}

```

Relative permeability is evaluated using the Van Genuchten–Mualem expression with a lower bound imposed to avoid vanishing hydraulic conductivity.

Listing 4.3 Relative permeability (Van Genuchten–Mualem model)

```

double FourNodeQuadUPSVG::getPermeability(double Se) const {
    constexpr double kr_min = 1e-6;
    Se = std::min(1.0, std::max(1e-12, Se));
    const double m = std::min(std::max(mVG, 1e-8), 1.0 - 1e-8);
    const double t = 0.5; // Mualem
    const double Se_t = std::pow(Se, t);
    const double Se_1m = std::pow(Se, 1.0/m);
    double kr = Se_t * std::pow(1.0 - std::pow(1.0 - Se_1m, m), 2.0);
    return std::min(std::max(kr, kr_min), 1.0);
}

```

4.5.3 Enhanced Mass Matrix `getMass()`

The following implementation of the mass matrix includes the contribution of the fluid and the porous skeleton and accounts for saturation-dependent compressibility:

Listing 4.4 Implementation of `getMass()`

```

const Matrix& FourNodeQuadUPSVG::getMass()
{
    K.Zero();
    this->shapeFunction();
    // --- Single-source saturation terms (at element center) ---
    const double pw = this->getElementPwCenter();
    const double n = std::min(std::max(getPorosity(0), 1e-8),
0.999999);

    const double Sr = calculateSaturation(pw);
    const double Se = std::min(1.0, std::max(0.0, (Sr - Sr_res) /
std::max(1.0 - Sr_res, 1e-12)));
    const double Sw = Sr;
}

```

```

const double Sw_eff = std::max(Sw, Sr_res + 1e-6);
const double dSe_dpw = derivativeSe(pw);           // 1/kPa
const double dSw_dpw = (1.0 - Sr_res) * dSe_dpw;    // 1/kPa
const double Kw = std::max(kc * n, 1e-12);          // kPa
// Hydraulic compressibility: cap only (no extra "fade")
double Cs = n * dSw_dpw;                           // 1/kPa
const double Cs_cap = 0.8 * (n * Sw_eff / Kw);      //
conservative physical cap
Cs = std::min(std::max(Cs, 0.0), Cs_cap);
// Final inverse storage (1/Q*): single floor
double invQ = Cs + n * Sw_eff / Kw;                // 1/kPa
// invQ = std::max(invQ, 1e-6);                    // optional
floor

// --- (u -- u) consistent mass ---
for (int i = 0, i1 = 0; i < 12; i += 3, ++i1) {
    for (int j = 0, j1 = 0; j < 12; j += 3, ++j1) {
        double Mij = 0.0;
        for (int m = 0; m < 4; ++m) {
            const double rhoMix = mixtureRho(m);
            Mij += dvol[m] * rhoMix * shp[2][i1][m] *
shp[2][j1][m];
        }
        K(i, j) += Mij;
        K(i + 1, j + 1) += Mij;
    }
}

//--- (p -- p) hydraulic storage ---
for (int i = 2; i < 12; i += 3) {
    const int i1 = (i - 2) / 3;
    for (int j = 2; j < 12; j += 3) {
        const int j1 = (j - 2) / 3;
        double Cij = 0.0;
        for (int m = 0; m < 4; ++m) {
            Cij += -dvol[m] * invQ * shp[2][i1][m] *
shp[2][j1][m];

```

```

        }
        K(i, j) += Cij;
    }
}
return K;
}

```

4.5.4 Modified Damping Matrix `getDamp()`

The following code shows the implementation of the full damping matrix, which includes classical Rayleigh damping, coupling terms, and saturation-dependent permeability:

Listing 4.5 Full implementation of `getDamp()`

```

const Matrix& FourNodeQuadUPSVG::getDamp()
{
    static Matrix Kdamp(12, 12);
    Kdamp.Zero();
    // --- Rayleigh damping contributions ---
    if (betaK != 0.0) Kdamp.addMatrix(1.0,
this->getTangentStiff(), betaK);
    if (betaK0 != 0.0) Kdamp.addMatrix(1.0,
this->getInitialStiff(), betaK0);
    if (betaKc != 0.0 && Kc != nullptr) Kdamp.addMatrix(1.0, *Kc,
betaKc);
    // --- Mass-proportional damping (uses current mass matrix) ---
    if (alphaM != 0.0) {
        this->getMass(); // fills K with the mass terms
        for (int i = 0; i < 12; i += 3) {
            Kdamp(i, i) += alphaM * K(i, i);
            Kdamp(i + 1, i + 1) += alphaM * K(i + 1, i + 1);
        }
    }
    // Shape functions, volumes, etc.
    this->shapeFunction();
    // --- Coupling blocks (Q and Q^T) ---
    for (int i = 0; i < 12; i += 3) {
        const int iNode = i / 3;

```

```

    for (int j = 2; j < 12; j += 3) {
        const int jNode = (j - 2) / 3;
        for (int m = 0; m < 4; ++m) {
            const double coeff = -dvol[m];
            const double Nix = shp[0][iNode][m];
            const double Niy = shp[1][iNode][m];
            const double Nj = shp[2][jNode][m];
            Kdamp(i, j) += coeff * Nix * Nj;
            Kdamp(i + 1, j) += coeff * Niy * Nj;
        }
        // symmetry
        Kdamp(j, i) = Kdamp(i, j);
        Kdamp(j, i + 1) = Kdamp(i + 1, j);
    }
}

// --- Relative permeability kr(Se) at element center (single
definition) ---
const double pw_elem = this->getElementPwCenter();
const double Sr_elem = calculateSaturation(pw_elem);
double Se_elem = (Sr_elem - Sr_res) / std::max(1.0 - Sr_res,
1e-12);
Se_elem = std::min(1.0, std::max(0.0, Se_elem));

const double kr_eff = getPermeability(Se_elem);
// --- Permeability block (p-p) with k = kr_eff * k_sat ---
const double kx = kr_eff * perm[0];
const double ky = kr_eff * perm[1];
for (int i = 2; i < 12; i += 3) {
    const int iNode = (i - 2) / 3;
    for (int j = 2; j < 12; j += 3) {
        const int jNode = (j - 2) / 3;
        for (int m = 0; m < 4; ++m) {
            const double Nix = shp[0][iNode][m];
            const double Njx = shp[0][jNode][m];
            const double Niy = shp[1][iNode][m];
            const double Njy = shp[1][jNode][m];

```



```

        Kdamp(i, j) += -dvol[m] * (kx * Nix * Njx + ky * Niy
* Njy);
    }
}
}
K = Kdamp;
return K;
}

```

4.5.5 Updated Force Vector `getResistingForce()`

The resisting vector now accounts for drag force due to unsaturated flow:

Listing 4.6 Full implementation of `getResistingForce()`

```

const Vector& FourNodeQuadUPSVG::getResistingForce()
{
    P.Zero();
    this->shapeFunction();

    // --- (1) Solid effective stresses + body forces ---
    for (int m = 0; m < 4; ++m) {
        const Vector& sigma = theMaterial[m]->getStress();
        const double sxx = sigma(0);
        const double syy = sigma(1);
        const double sxy = sigma(2);

        for (int a = 0, ia = 0; a < 4; ++a, ia += 3) {
            P(ia)      += dvol[m] * (shp[0][a][m] * sxx + shp[1][a][m]
* sxy);
            P(ia + 1) += dvol[m] * (shp[1][a][m] * syy + shp[0][a][m]
* sxy);

            const double rhoMix = mixtureRho(m);
            const double* bVec = (applyLoad == 0) ? b : appliedB;
            P(ia)      -= dvol[m] * shp[2][a][m] * rhoMix * bVec[0];
            P(ia + 1) -= dvol[m] * shp[2][a][m] * rhoMix * bVec[1];
        }
    }
}

```

```

}

// --- (2) Fluid part: Darcy term with a single kr definition
// (same as getDamp) ---
const double pw_elem = this->getElementPwCenter();
const double Sr_elem = calculateSaturation(pw_elem);
double Se_elem = (Sr_elem - Sr_res) / std::max(1.0 - Sr_res,
1e-12);
Se_elem = std::min(1.0, std::max(0.0, Se_elem));

const double kr_eff = getPermeability(Se_elem);

const double kx    = kr_eff * perm[0];
const double ky    = kr_eff * perm[1];
const double rho_f = rho;

for (int a = 0, ia = 0; a < 4; ++a, ia += 3) {
    const double* bVec = (applyLoad == 0) ? b : appliedB;
    for (int m = 0; m < 4; ++m) {
        P(ia + 2) += dvol[m] * rho_f * (
            kx * bVec[0] * shp[0][a][m] + // dN/dx
            ky * bVec[1] * shp[1][a][m] // dN/dy
        );
    }
}

// --- (3) External surface loads and hydraulic boundary terms ---
if (pressure != 0.0) {
    P.addVector(1.0, pressureLoad, -1.0);
}
P.addVector(1.0, Q, -1.0);

return P;
}

```

4.5.6 Tcl Commands

The FourNodeQuadUPSVG element, whose implementation has been described in this chapter, can be invoked through the OpenSees Tcl interpreter using the following command:

```

1 element QuadUPSVG $eleTag \
2   $iNode $jNode $kNode $lNode \
3   $thk $matTag \
4   $bulk $rho_f $perm_x $perm_y \
5   $alphaVG $mVG $Sr_res \
6   $k_a $k_s $e \
7   <$b1 $b2 $pressure>

```

Here, `$eleTag` denotes the element tag, and `$iNode`, `$jNode`, `$kNode`, and `$lNode` are the tags of the four corner nodes (the element is developed for plane-strain conditions). Each node has three degrees of freedom: two solid displacement components (u_x, u_y) and one pore-pressure degree of freedom (p). The element is formulated within the standard OpenSees fully-coupled u – p framework and is based on the FourNodeQuadUP formulation (simplified Biot theory), extended in this work to incorporate unsaturated behaviour through a Van Genuchten-type soil-water retention relationship and a relative-permeability correction.

The mandatory input parameters define the geometry, constitutive response, hydraulic properties, and unsaturated parameters. In particular:

- `$thk` is the element thickness t ;
- `$matTag` is the tag of the associated NDMaterial;
- `$bulk` is the bulk modulus parameter k_c employed in the storage term (compressibility contribution);
- `$rho_f` is the fluid density ρ_f ;
- `$perm_x` and `$perm_y` are the saturated intrinsic permeabilities k_x and k_y .

The unsaturated behaviour is controlled by the Van Genuchten parameters `$alphaVG`, `$mVG`, and `$Sr_res`, where α_{VG} governs the air-entry suction scale, m_{VG} controls the curve shape, and $S_{r,res}$ is the residual saturation. These parameters are stored at the element level and are also passed to the material through the dedicated interface method `setTrialVGParameters` to ensure consistent hydro–mechanical coupling across the element–material boundary.

In addition, the parameters k_a and k_s correspond to the bulk moduli of air and water used in the effective fluid compressibility evaluation, while e is the initial void ratio, used to compute the porosity $n = e/(1 + e)$ required by the storage formulation.

The optional parameters enclosed within angle brackets $\langle \rangle$ allow the definition of body forces and surface pressure:

- b_1, b_2 : constant body force components (e.g., gravity direction and magnitude),
- $pressure$: uniform boundary pressure applied along the element edges (consistent with the standard OpenSees pressure load convention for quadrilateral elements).

Element outputs. Among the different outputs available for this element, stress and strain quantities at Gauss points can be recorded using standard OpenSees recorder Element commands. In addition, specific unsaturated/hydraulic quantities were implemented and can be obtained via the following requests:

1	recorder Element -ele \$eleTag -time -file flux.out	flux
2	recorder Element -ele \$eleTag -time -file SrAvg.out	SrAvg
3	recorder Element -ele \$eleTag -time -file kxy.out	k
4	recorder Element -ele \$eleTag -time -file kr.out	kr

In the above, flux returns the Darcy flux components at the element center (q_x, q_y) , SrAvg returns the average saturation (evaluated at the element center pressure), k returns the effective permeability components (k_x, k_y) including the relative-permeability correction under unsaturated conditions, invQ returns the storage-related quantity used in the u - p mass term, and kr returns the effective relative permeability factor k_r .

Finally, it is noted that the element computes the pore pressure at the element center from the nodal pressure DOFs and evaluates suction as $s = \max(-p_w, 0)$, with saturation computed using the Van Genuchten relationship. These values are then consistently propagated to the constitutive model at each Gauss point through setTrialSaturation, setTrialSuction, and setTrialVGParameters, while preserving the modular structure of OpenSees.

Chapter 5

Verification under Static and Transient Hydro-Mechanical Conditions

5.1 Introduction

The new QuadUPSVG finite element, hereafter referred to as New Element, whose formulation and implementation in OpenSees were widely discussed in Chapter 4, was then subjected to a verification process to assess its ability in reproducing physical problems involving partially saturated soils. This verification should be carried out under both static and dynamic conditions. As a consequence, this chapter focuses on the verification of the proposed formulation under static and transient hydro–mechanical conditions. The objective is to assess the correctness, robustness, and physical consistency of the finite element when subjected to saturated and partially saturated boundary value problems.

The verification strategy is designed to progressively evaluate the different components of the coupled formulation. Particular attention is given to the interpolation of primary variables, the enforcement of hydraulic and mechanical boundary conditions, the transfer of stress between the solid skeleton and the pore fluid, and the numerical stability of transient diffusion processes.

To clearly distinguish the finite element formulation from the nonlinearity due to the constitutive behaviour, the benchmarks are conducted using constitutive models of progressively increasing complexity. In a first stage, a linear elastic model is adopted to isolate the performance of the discretized governing equations and the global hydro–mechanical coupling structure.

Subsequently, the ManzariDafaliasIsoHard model developed in Chapter 3 is employed to evaluate the fully coupled response of partially saturated soils, where mechanical

behaviour, suction evolution, and degree of saturation are simultaneously governed within a unified hydro–mechanical framework.

The numerical results obtained with the QuadUPSVG element are compared against established reference solutions, including the FourNodeQuadUP element available in the OpenSees library, i.e., the classical quadrilateral four-node element with coupled u-p hydro–mechanical formulation, analytical and analytical–numerical solutions, and benchmark results reported in the literature. Through this systematic comparison, the present chapter provides a comprehensive verification of the proposed formulation in static and transient hydro–mechanical regimes, establishing a reliable foundation for subsequent dynamic analyses that will be discussed in Chapter 6.

5.2 Verification Strategy

The verification strategy adopted in this chapter is based on a sequence of numerical benchmarks designed to isolate and evaluate specific aspects of the coupled hydro–mechanical formulation. These include the treatment of hydraulic and mechanical boundary conditions, the transfer of stresses between the solid skeleton and the pore fluid, the evolution of pore–water pressure and degree of saturation, and the numerical stability of the formulation under transient and dynamic loading.

Each benchmark is analyzed using the QuadUPSVG and QuadUP elements, and evaluated through direct comparison with reference solutions, allowing the performance of the formulation to be assessed in a controlled and systematic manner before considering more complex geotechnical boundary value problems.

5.3 Mono-dimensional consolidation

As an initial verification case, a classical one-dimensional consolidation benchmark is analyzed. The latter is also reported as a reference case for the OpenSees users [98]. This example was selected as a baseline test to verify the correct implementation of the coupled governing equations under fully saturated conditions and to ensure consistency with the reference QuadUP formulation.

The problem consists of a laterally constrained vertical soil column idealized as a two-dimensional domain of unit width and total height $H = 10\text{m}$, exhibiting one-dimensional consolidation behavior (see Figure 5.1). The column was discretized using a structured mesh of 100 vertically stacked quadrilateral QuadUP elements, resulting a total of 202 nodes. To enforce one-dimensional deformation, both horizontal and vertical displacement

degrees of freedom were coupled between the left and right nodes at each elevation, through equalDOF constraints. Three degrees of freedom (u_x, u_y, p_w) were assigned to each node, corresponding to horizontal displacement, vertical displacement, and pore-water pressure, respectively. The behaviour of the soil skeleton was modeled using a linear elastic constitutive law (LinearElastic). The elastic and hydraulic properties adopted in the analysis are summarized in Table 5.1.

Table 5.1 Elastic and hydraulic properties adopted

Property	Symbol	Value	Unit
Bulk modulus	K	2.2×10^5	kPa
Poisson's ratio	ν	0.495	–
Young's modulus	E	6.6×10^3	kPa
Solid density (NDMaterial input)	ρ_s	1.7	t/m ³
Fluid bulk modulus	K_f	5.5×10^6	kPa
Water density	ρ_w	1.0	t/m ³
Gravity acceleration	g	10.0	m/s ²
Saturated permeability (horizontal)	$k_{sat,x}$	4.0×10^{-4}	m/s
Saturated permeability (vertical)	$k_{sat,y}$	4.0×10^{-4}	m/s

The base of the column was fixed against both horizontal and vertical displacements, while lateral boundaries were restrained in the horizontal direction only, allowing vertical settlement.

The consideration of both single- and double-drainage configurations allows assessing the correct enforcement of hydraulic boundary conditions and the consistency of the coupled hydro-mechanical formulation under distinct pore-pressure dissipation mechanisms. Since drainage conditions directly govern the rate of consolidation and the spatial distribution of excess pore-water pressures, reproducing the different consolidation responses associated with each configuration provides a stringent verification of the flow equation and its coupling with the mechanical equilibrium equation.

Drainage conditions were imposed by prescribing zero pore-water pressure ($p_w = 0$) at the top boundary for the single-drainage case, and at both the top and bottom boundaries for the double-drainage case. Uniform vertical settlement along the column was ensured through the application of equalDOF constraints to the nodes located at the same height. The recorded responses include pore-water pressure dissipation, effective stress evolution, and settlement history. In addition to the boundary conditions described above, two alternative loading strategies were considered in order to further verify the robustness of the implementation. The first strategy, consistent with the one reported in McGann and Arduino [98], consists of applying the external load directly to the soil skeleton in the form of a uniform vertical

surcharge of 1000 kPa at the top boundary. Since the model width and out-of-plane thickness are equal to unity, this corresponds to a uniformly distributed surface load of the same magnitude, equally shared between the two top nodes. The second strategy considers the application of the load indirectly through the pore fluid, by prescribing an increment of pore-water pressure $\Delta p_w = 1000 \text{ kPa}$ at the drainage boundary. In practice, this was implemented by applying the load to the third nodal degree of freedom ($DOF = 3$), which in the QuadUP formulation corresponds to the pore-water pressure variable.

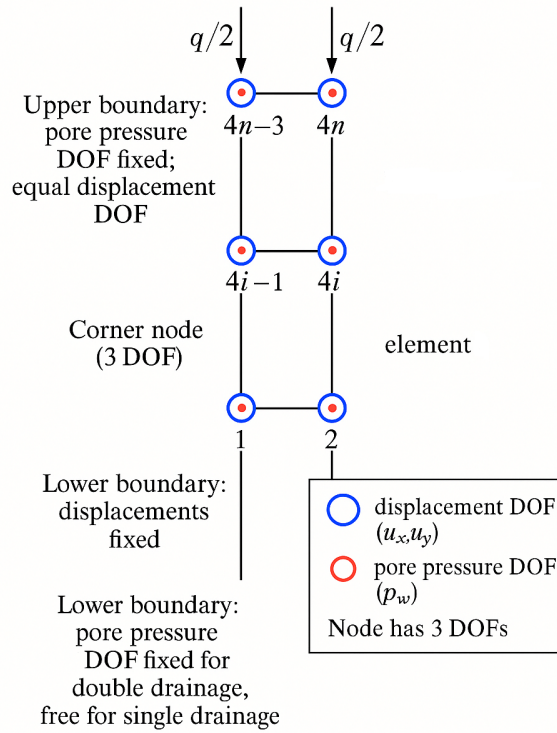


Fig. 5.1 Soil column geometry and boundary conditions adopted for the one-dimensional consolidation benchmark.

Both loading strategies were examined under identical constraint conditions for single- and double-drainage configurations, consistent with the original benchmark example in OpenSees. All mechanical and hydraulic properties were kept identical to those reported in Table 5.1. Before discussing the results, it is important to clarify that the two loading strategies considered in this benchmark are not expected to produce an identical transient response. When the external load is applied directly to the soil skeleton, the initial excess pore-water pressure is generated through the coupled undrained response of the solid–fluid mixture. In contrast, when the load is introduced through prescribed increments of pore-water

pressure, the perturbation is imposed directly on the hydraulic degree of freedom. Therefore, although both procedures may lead to comparable final consolidation states, the intermediate pore-pressure, stress-transfer, and settlement responses can differ. These differences are a natural consequence of the coupled formulation, in which the hydraulic conductivity, fluid bulk modulus, porosity, and compressibility of the soil skeleton jointly control the rate and mechanism of load transfer between the pore fluid and the solid phase. Consequently, the discrepancies observed between loading applied to the soil skeleton and loading imposed through the pore-pressure degree of freedom should not be interpreted as inconsistencies of the formulation, but rather as the expected response of a coupled hydro-mechanical system subjected to different loading paths. This distinction will also be relevant in the following results where similar loading conditions are imposed.

Figures 5.2a and 5.2b present the pore-water pressure profiles over time obtained by assigning the external load directly to the soil skeleton under single-drainage conditions, using the QuadUP element and the New Element, respectively. Obviously, the New Element is considered under saturated conditions, i.e., with a unit degree of saturation. A vertical load of 1000 kPa was applied, resulting in an initial maximum excess pore-water pressure of approximately 1000 kPa, which is consistent with a fully saturated undrained response. In both cases, pore-water pressure dissipates progressively from the drained boundary toward the interior of the soil column, with smooth and monotonic profiles evolving consistently over time. The dissipation rate is primarily governed by the adopted hydraulic conductivity. The close correspondence between the two sets of results indicates that the proposed element reproduces the hydraulic response of the reference formulation when the load is applied to the soil skeleton, this behaviour being consistent for both elements, which exhibit the same peak pore pressure and nearly identical dissipation patterns.

Figures 5.3a and 5.3b show the pore-water pressure profiles obtained when the load is applied through prescribed pore-water pressure increments under the same single-drainage conditions. In this case, the initial pore-pressure distribution is uniform along the depth of the column, with a magnitude of approximately 1000 kPa, followed by a gradual dissipation from the drained boundary. Slight variations between the two elements can be observed at early stages, particularly near the drained boundary, which are attributed to the coupling governed by the fluid bulk modulus and hydraulic conductivity. Nevertheless, the overall dissipation trends and magnitude of pore pressures remain consistent, confirming that both formulations provide an equivalent consolidation response.

The transfer of load from the pore fluid to the soil skeleton under single-drainage conditions is examined through the stress profiles shown in Figures 5.4a and 5.4b for loading applied directly to the soil skeleton, and in Figures 5.5a and 5.5b when the load is ap-

plied through prescribed pore-water pressure increments. In all cases, the stress profiles obtained with the New Element are the same with those obtained with the QuadUP formulation, demonstrating a consistent redistribution of stress as pore-water pressure dissipates. These results indicate that the effective stress response is independent of the loading strategy, provided that the coupled formulation is implemented consistently.

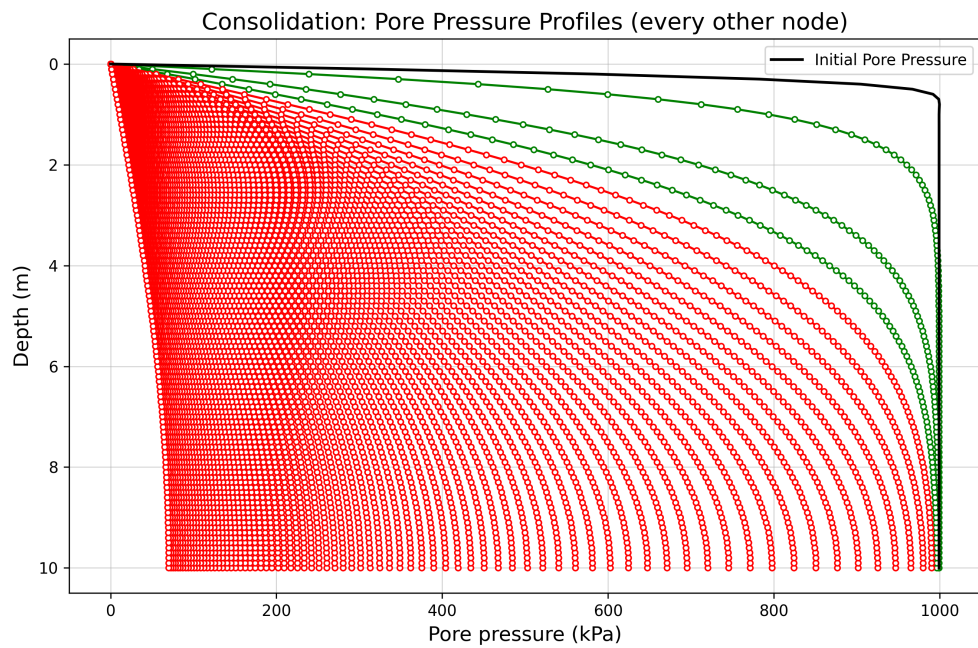
The global response of the soil column under single-drainage conditions is further assessed by examining the surface settlement histories shown in Figures 5.6a and 5.6b. Both formulations exhibit the characteristic time-dependent settlement behavior associated with one-dimensional consolidation, with the same settlement magnitudes and temporal evolution. The close agreement between the two curves confirms that the proposed element accurately reproduces the overall mechanical response of the system.

The effect of drainage conditions is illustrated in Figures 5.7a and 5.7b, which show the pore-water pressure profiles obtained under double-drainage conditions when the load is applied directly to the soil skeleton using the QuadUP and New Element formulations, respectively. Once more, the New Element is considered under fully-saturated conditions. Compared to the single-drainage case, pore-water pressure dissipates more rapidly, and the profiles exhibit a symmetric distribution with respect to the mid-height of the soil column.

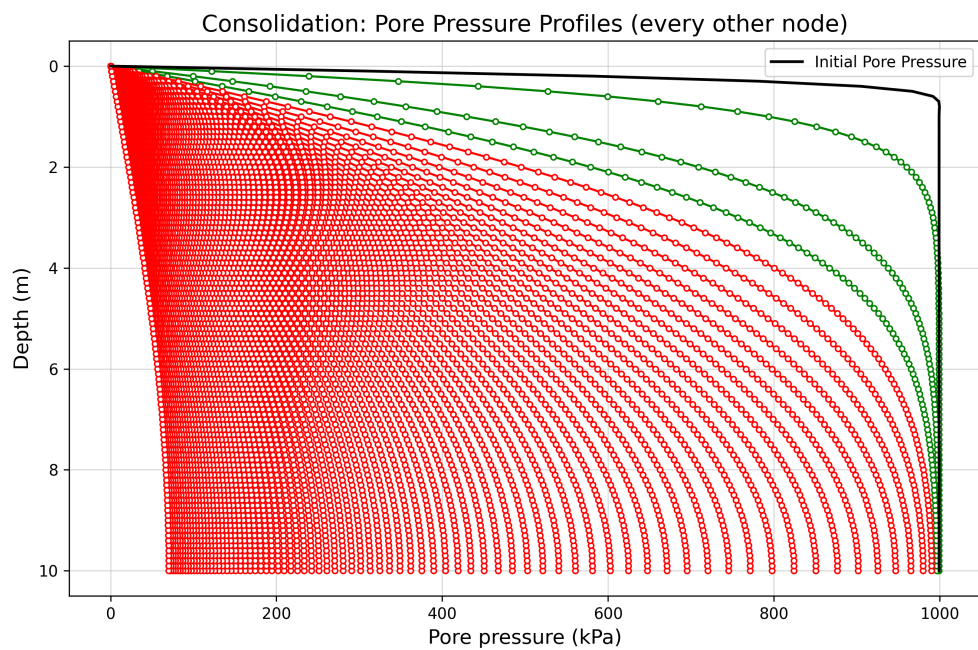
Similar observations can be made from Figures 5.8a and 5.8b, where the load is applied through prescribed pore-water pressure increments, further confirming the consistency of the hydraulic response under double-drainage conditions.

The corresponding pore-pressure profiles under double-drainage conditions are presented in Figures 5.9a– 5.10b for both loading strategies and element formulations. A vertical load of 1000 kPa was applied, producing an initial maximum excess pore-water pressure of approximately 1000 kPa at the mid-height of the column, consistent with symmetric undrained conditions. Due to the double-drainage configuration, dissipation occurs simultaneously toward the top and bottom boundaries, resulting in symmetric profiles and a shorter consolidation time compared to the single-drainage case. The results obtained with the QuadUP and New Element formulations are identical, confirming that the coupled hydro–mechanical response is reproduced exactly under double-drainage conditions.

Similarly, when the load is applied through prescribed pore-water pressure increments (Figures 5.11a–5.11b), an initial uniform distribution of approximately 1000 kPa is imposed, followed by symmetric dissipation toward both drained boundaries. The responses obtained with both elements coincide entirely in magnitude and evolution, demonstrating full equivalence of the two implementations under saturated conditions.



(a) QuadUP (single drainage).



(b) New Element (single drainage).

Fig. 5.2 Pore-water pressure profiles under single-drainage conditions for loading applied to the soil skeleton.

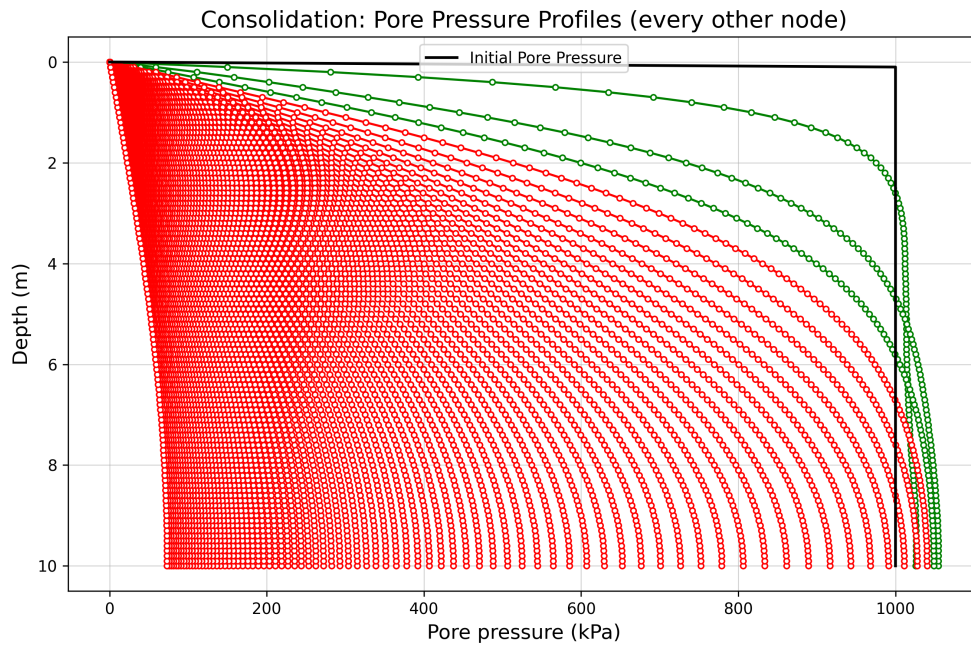
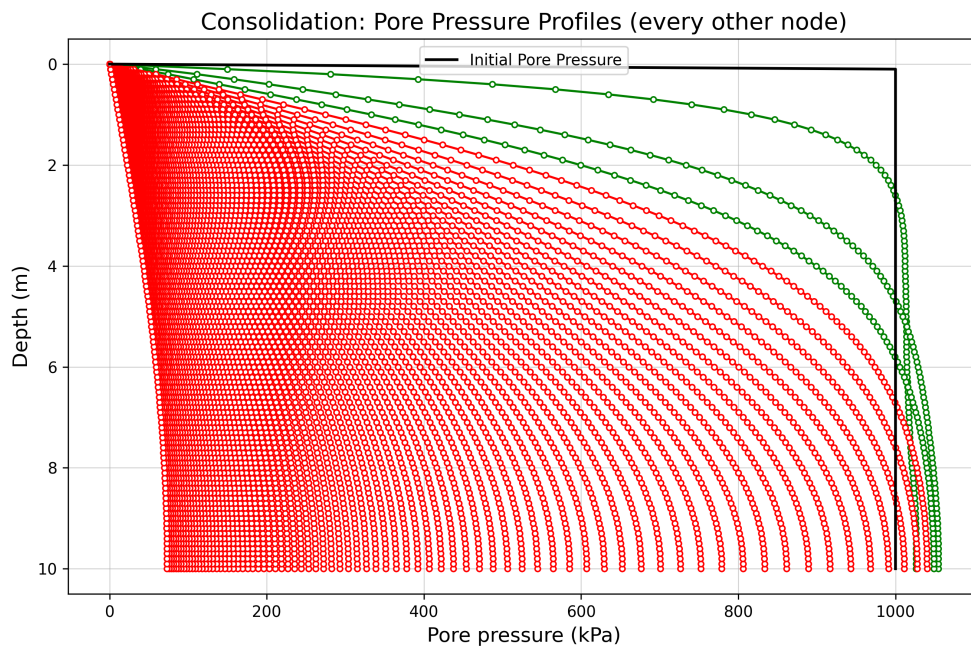
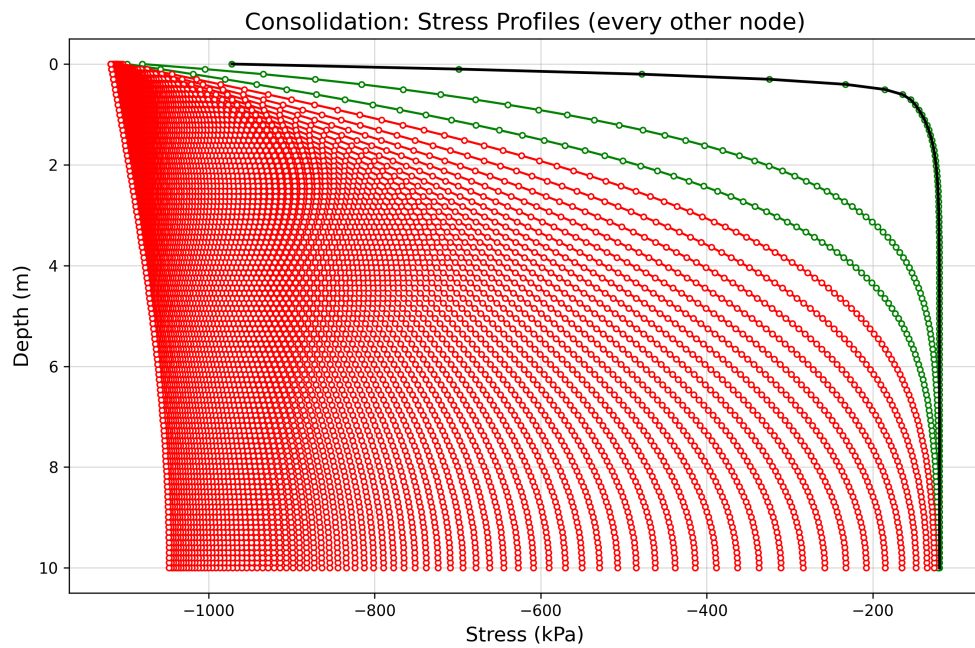
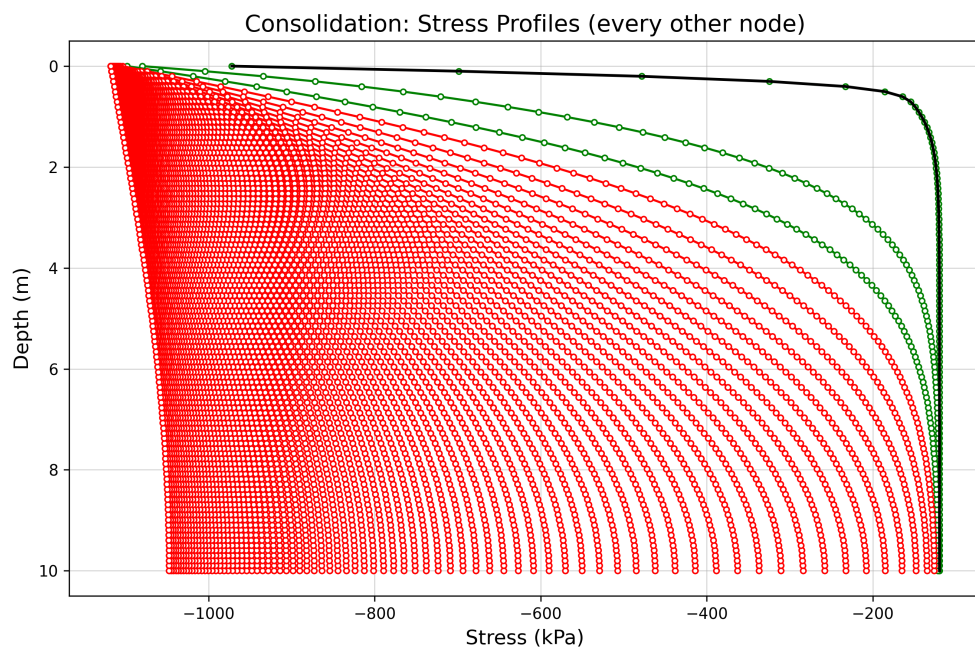
(a) QuadUP (single drainage, prescribed p_w).(b) New Element (single drainage, prescribed p_w).

Fig. 5.3 Pore-water pressure profiles for single-drainage conditions with loading applied through prescribed pore-water pressure increments.

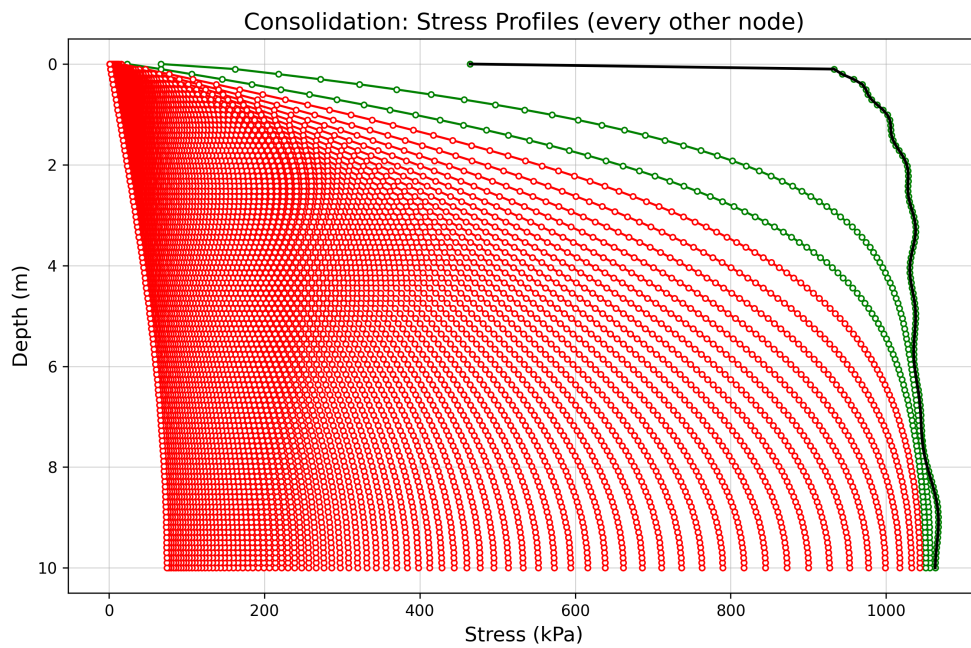


(a) QuadUP.

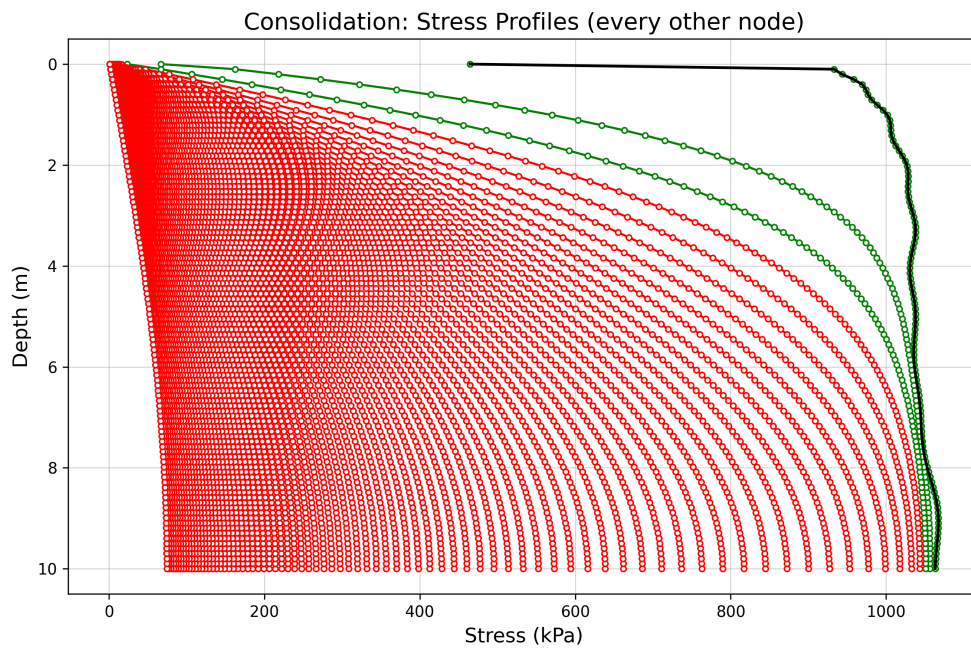


(b) New Element.

Fig. 5.4 Stress profiles under single-drainage conditions with loading applied directly to the soil skeleton.



(a) QuadUP.

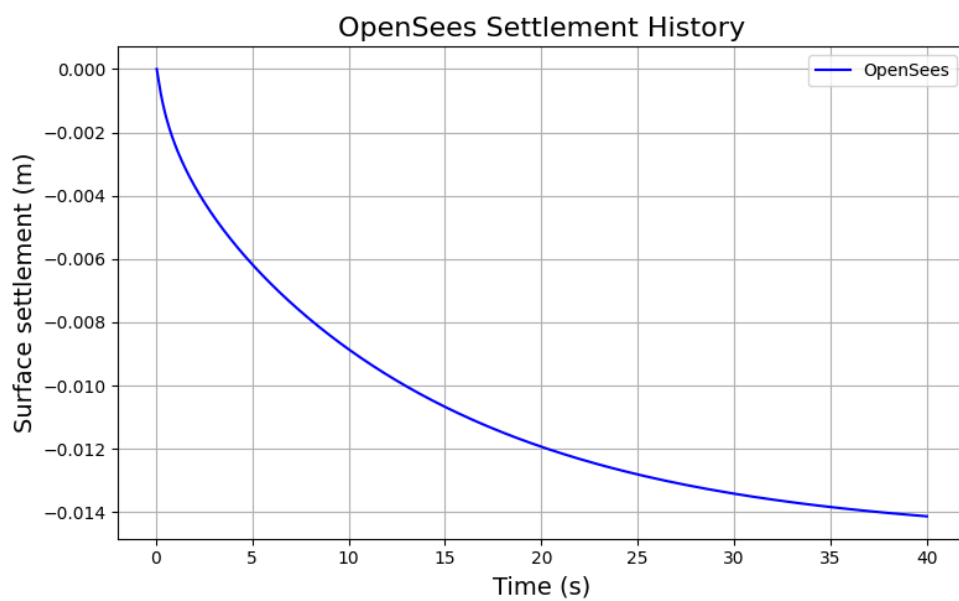


(b) New Element.

Fig. 5.5 Stress profiles under single-drainage conditions with loading applied through prescribed pore-water pressure increments.

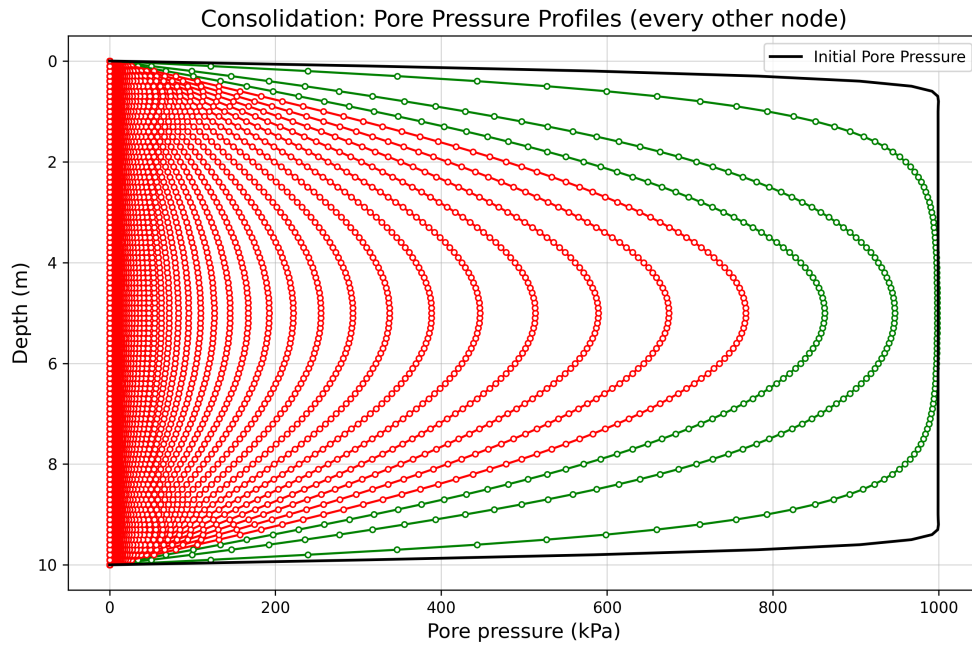


(a) QuadUP.

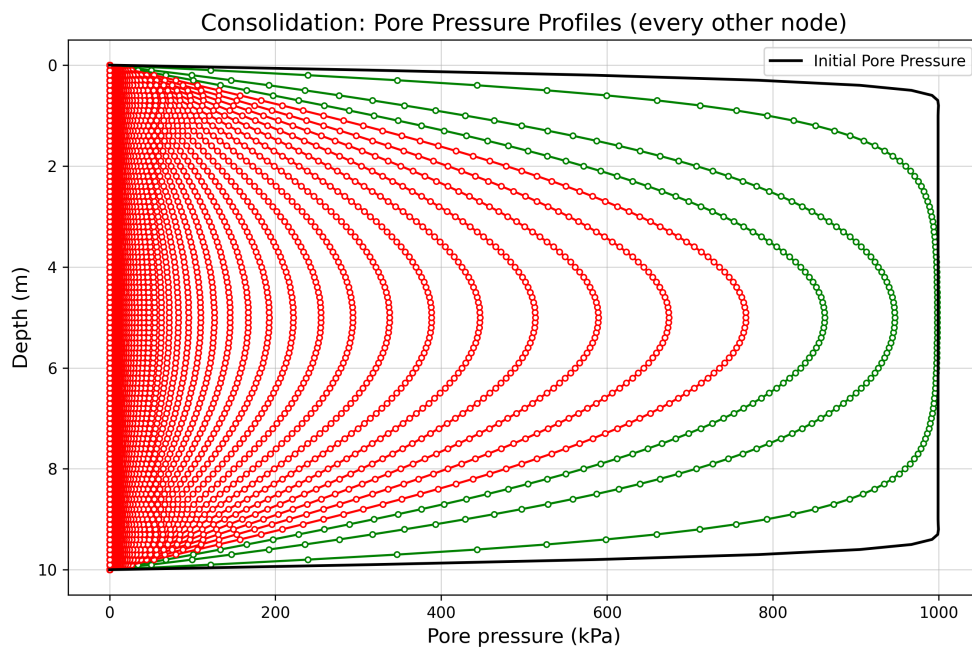


(b) New Element.

Fig. 5.6 Surface settlement histories under single-drainage conditions.

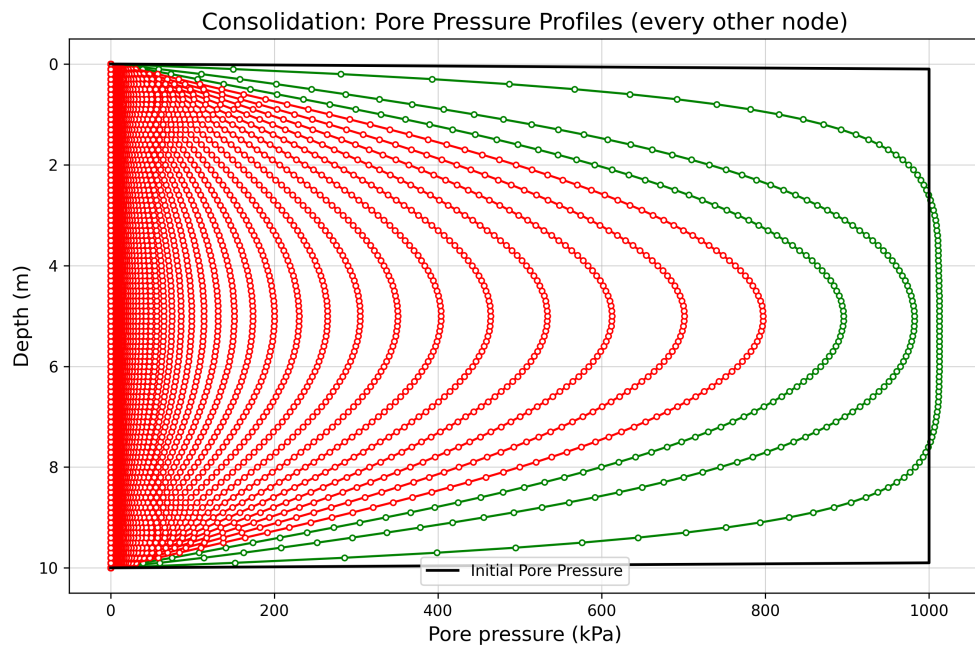


(a) QuadUP.

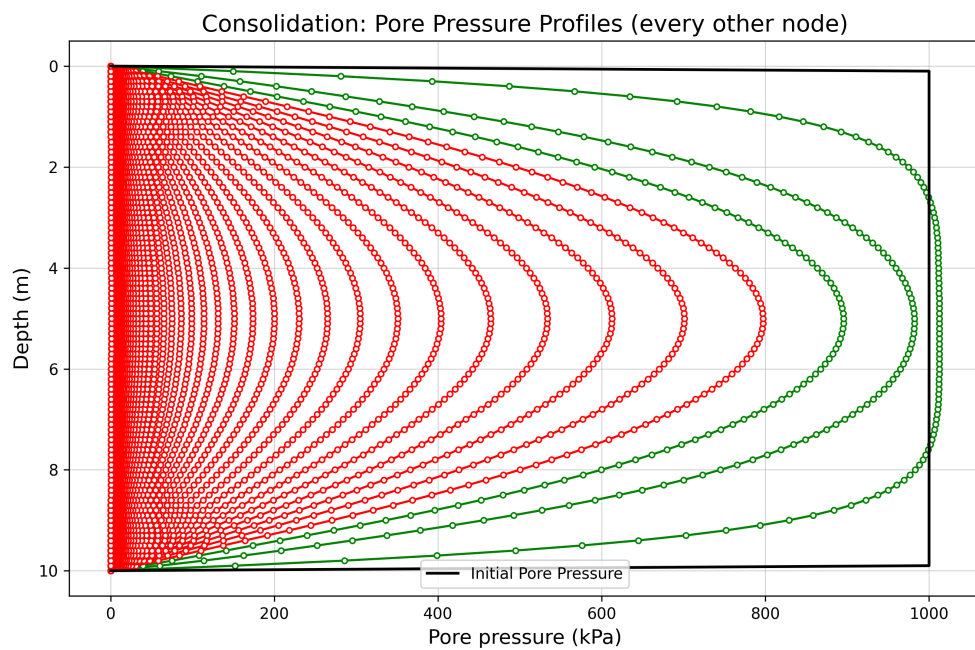


(b) New Element.

Fig. 5.7 Pore-water pressure profiles under double-drainage conditions with loading applied directly to the soil skeleton.

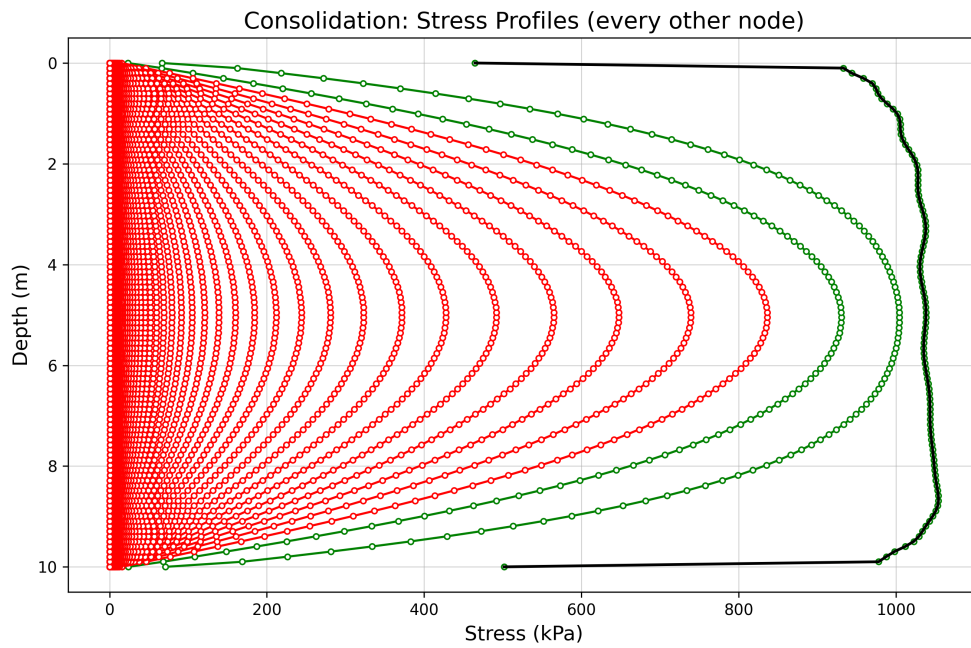


(a) QuadUP.

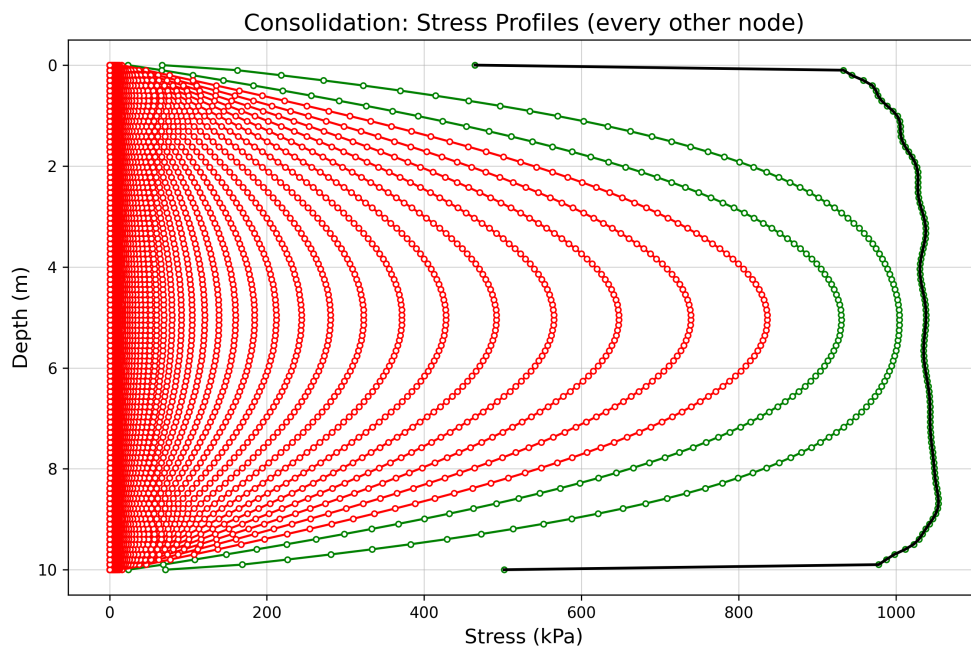


(b) New Element.

Fig. 5.8 Pore-water pressure profiles under double-drainage conditions with loading applied through prescribed pore-water pressure increments.

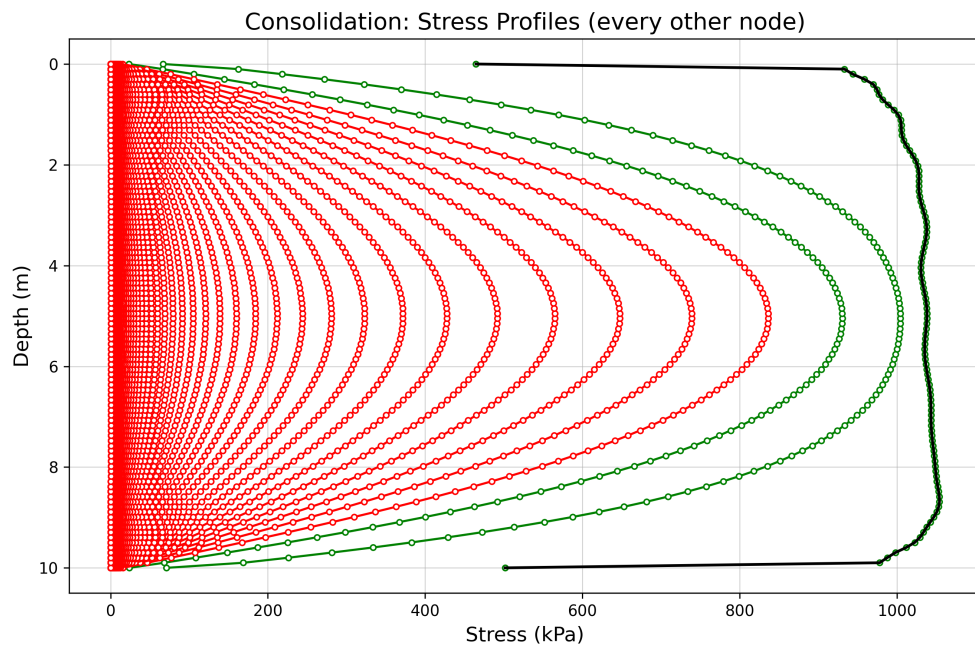


(a) QuadUP.

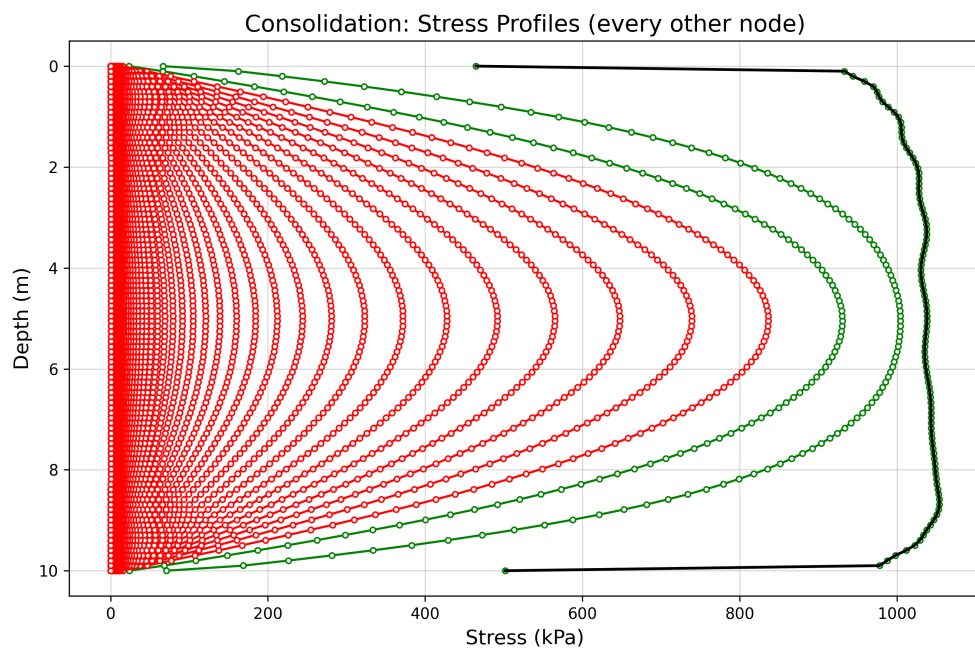


(b) New Element.

Fig. 5.9 Stress profiles under double-drainage conditions with loading applied directly to the soil skeleton.

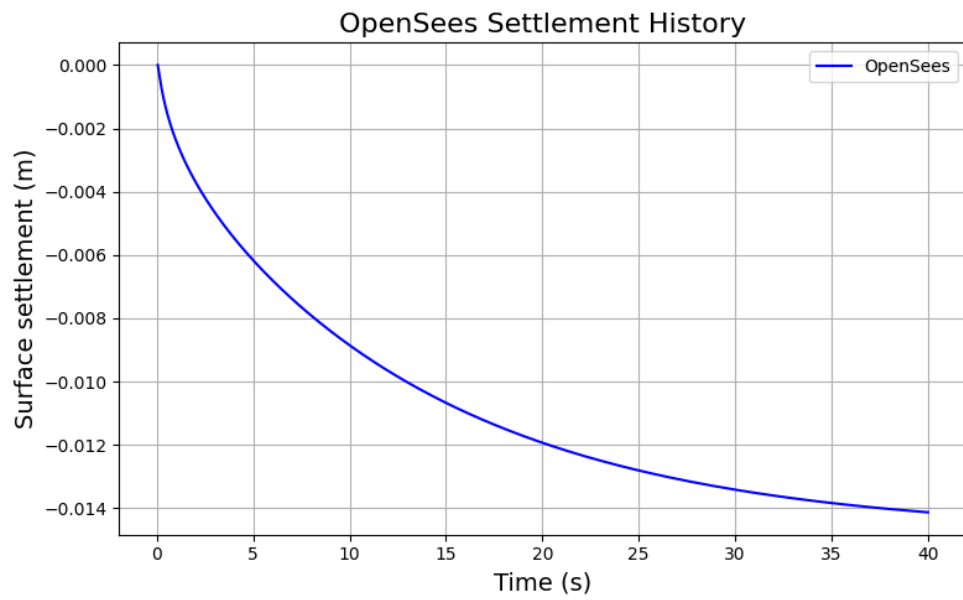


(a) QuadUP.

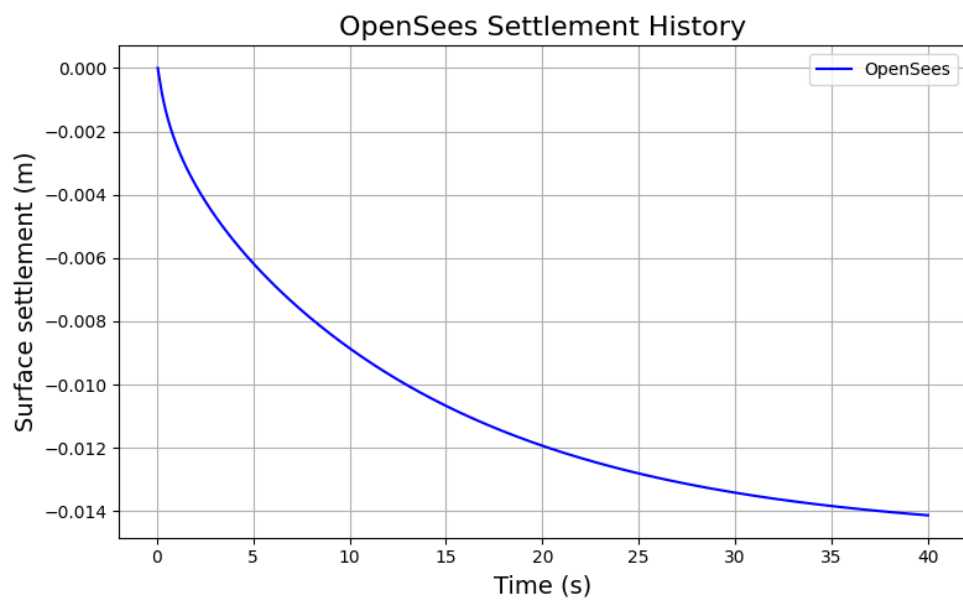


(b) New Element.

Fig. 5.10 Stress profiles under double-drainage conditions with loading applied through prescribed pore-water pressure increments.



(a) QuadUP.



(b) New Element.

Fig. 5.11 Surface settlement histories under double-drainage conditions.

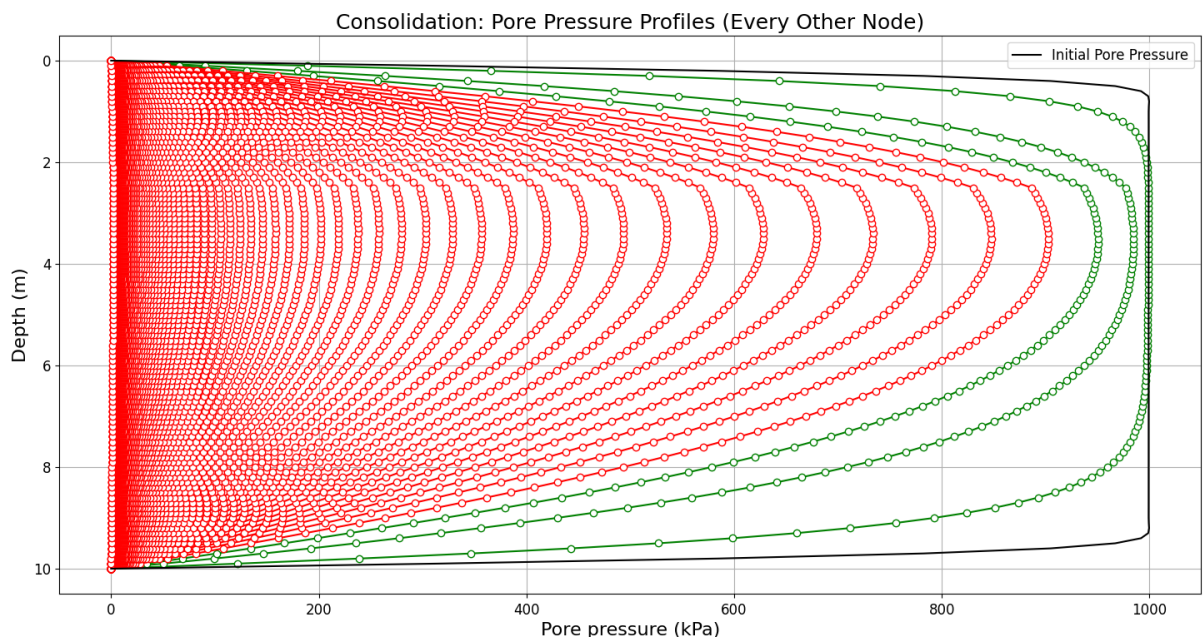
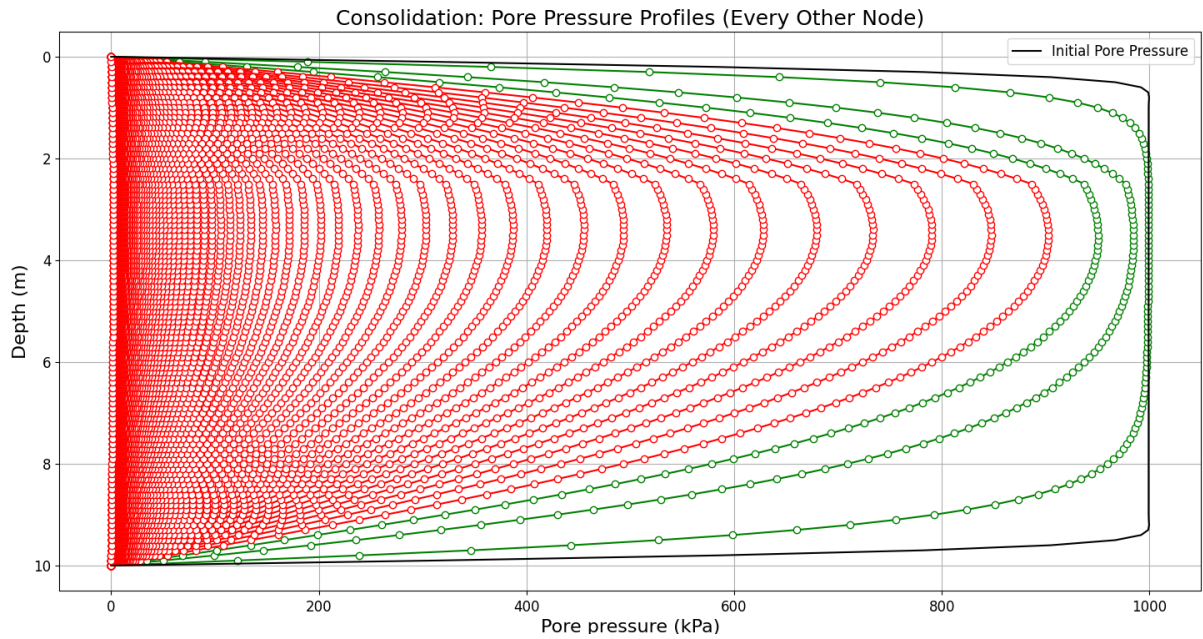


Fig. 5.12 Pore-water pressure profiles under double-drainage conditions for the layered soil configuration.

In addition to the homogeneous soil configuration, the double-drainage consolidation problem was also analyzed considering a two-layer soil column characterized by different hydraulic permeabilities. The mechanical properties of the two materials were kept identical

and correspond to those reported in Table 5.1. In contrast, the saturated permeability was assigned as $k_{sat,x} = k_{sat,y} = 1.0 \times 10^{-4}$ m/s in the upper layer and $k_{sat,x} = k_{sat,y} = 5.0 \times 10^{-5}$ m/s in the lower layer. This permeability contrast was introduced to evaluate the capability of the formulation to handle spatial variations in hydraulic properties within the same domain.

Figures 5.12a and 5.12b present the pore-water pressure profiles obtained for the layered configuration using the QuadUP and New Element formulations, respectively. In both cases, the results clearly reflect the influence of the permeability contrast, with different dissipation rates observed within each layer and a nonuniform evolution of pore-water pressure along the depth of the soil column. The change in slope of the pressure profiles across the layer interface highlights the effect of hydraulic heterogeneity on the consolidation process.

A close comparison between Figures 5.12a and 5.12b shows that the responses obtained with the New Element are the same produced by the reference QuadUP formulation throughout the entire domain. The agreement is observed in both the shape of the pore-water pressure profiles and their temporal evolution, indicating that the proposed element consistently reproduces the reference response even in the presence of layered permeability.

Overall, the results presented in Figures 5.2a through 5.12b demonstrate that the New Element consistently reproduces the reference QuadUP response for the classical one-dimensional consolidation problem under both single- and double-drainage conditions and under fully-saturated conditions. The close agreement observed in pore-water pressure dissipation, stress evolution, and surface settlement, including cases involving layered permeability, verifies the correct implementation of the governing equations and the associated hydro-mechanical coupling. This benchmark, therefore, provides a robust verification of the proposed finite element formulation and establishes a reliable basis for subsequent applications involving partially saturated conditions and more complex geotechnical boundary value problems.

In addition to the numerical verification carried out through the comparison between the QuadUP and the proposed New Element formulations, an independent reference solution was also considered in order to further assess the accuracy of the implemented model. Specifically, the one-dimensional consolidation problem was analyzed using an analytical–numerical approach based on Terzaghi’s classical consolidation theory.

The finite difference discretization outlined above provides a reference solution to Terzaghi’s one-dimensional consolidation equation. This analytical–numerical scheme was employed to generate theoretical pore-water pressure dissipation profiles under both single- and double-drainage conditions. These solutions serve as an independent benchmark against which the numerical results obtained from the OpenSees simulations can be evaluated.

By comparing the excess pore-water pressure distributions obtained using the New Element, the reference QuadUP formulation, and the finite difference solution, it is possible to assess the consistency and accuracy of the proposed implementation across different drainage conditions and consolidation times. This comparison allows the numerical results to be interpreted not only in a relative sense (with respect to an existing finite element formulation), but also with respect to the expected theoretical response of the one-dimensional consolidation problem.

The corresponding analytical–numerical and finite element results are presented in the following figures. For completeness, the one-dimensional consolidation equation proposed by Terzaghi [140] is recalled below:

$$\frac{\partial u}{\partial t} = c_v \frac{\partial^2 u}{\partial z^2} \quad (5.1)$$

where:

- $u(z, t)$ is the excess pore water pressure,
- c_v is the coefficient of consolidation [m^2/s],
- z is the vertical coordinate,
- t is time.

Regarding the spatial and temporal discretization, it was assumed a discrete mesh with spatial step Δz and time step Δt . For each node i and time level n , the following auxiliary coefficients are defined:

$$\alpha_i = \frac{\Delta z_i}{\Delta t} \quad (5.2)$$

$$\beta_i = \frac{k_i \cdot \Delta z_i}{\Delta t^2} \quad (5.3)$$

where k_i is the hydraulic conductivity of layer i .

In this way, the pore pressure update scheme was done as follows.

$$u_i^{n+1} = u_i^n + (A_i \cdot u_{i-1}^n - 2B_i \cdot u_i^n + C_i \cdot u_{i+1}^n) \quad (5.4)$$

Where:

$$A_i = \frac{2\beta_i\beta_{i+1}}{\beta_{i+1}\alpha_i + \beta_i\alpha_{i+1}} \cdot \alpha_i \quad (5.5)$$

$$B_i = \frac{2\beta_i\beta_{i+1}}{\beta_{i+1}\alpha_i + \beta_i\alpha_{i+1}} \cdot \frac{\alpha_i + \alpha_{i+1}}{2} \quad (5.6)$$

$$C_i = \frac{2\beta_i\beta_{i+1}}{\beta_{i+1}\alpha_i + \beta_i\alpha_{i+1}} \cdot \alpha_{i+1} \quad (5.7)$$

The finite difference scheme described above corresponds to an explicit numerical approximation of Terzaghi's diffusion-type one-dimensional consolidation equation, extended to account for layered soil profiles with spatially variable hydraulic properties. This approach provides an analytical–numerical reference solution that can be used to directly assess the accuracy of the proposed New Element formulation.

To this end, selected time instants representative of the consolidation process were extracted from the OpenSees simulations performed using the New Element, as well as from the corresponding finite difference solution. These time instants represent both early and advanced stages of pore-water pressure dissipation, allowing the temporal evolution of the pressure profiles to be compared under identical drainage and material conditions.

Figure 5.13 presents a direct comparison between the excess pore-water pressure profiles obtained using the New Element in OpenSees and those computed with the analytical–numerical finite difference scheme at the selected times. The comparison is performed in terms of pore-water pressure distribution along the depth of the soil column, providing an independent verification of the ability of the proposed element to reproduce the theoretical consolidation response. It should be noted that the analytical–numerical solution is based on a decoupled formulation, in which the fluid flow problem and the response of the soil skeleton are not fully coupled. In contrast, the proposed New Element accounts for the coupled hydro-mechanical interaction between the pore fluid and the deformable soil skeleton. Therefore, the small differences observed between both solutions can be attributed to this difference in the adopted formulation. Overall, the agreement remains very good, confirming that the proposed element is able to reproduce the expected consolidation behaviour while retaining the coupled nature of the problem.

In addition to the linear elastic analyses presented above, the same one-dimensional consolidation problem was also investigated using the elasto-plastic constitutive model `ManzariDafaliasIsoHard`, previously described in Chapter 3.

The objective of this additional analysis was to verify the fully saturated response of the coupled formulation when nonlinear stress integration is involved, and to assess the consistency of the finite element implementation in combination with the constitutive model.

To this end, the consolidation benchmark was performed using both the reference QuadUP element and the proposed New Element, each coupled with the implemented

ManzariDafaliasIsoHard material. Hydraulic properties and boundary conditions were kept identical to those defined for the elastic case, so that the comparison isolates the combined effect of the finite element formulation and the constitutive integration algorithm under saturated conditions.

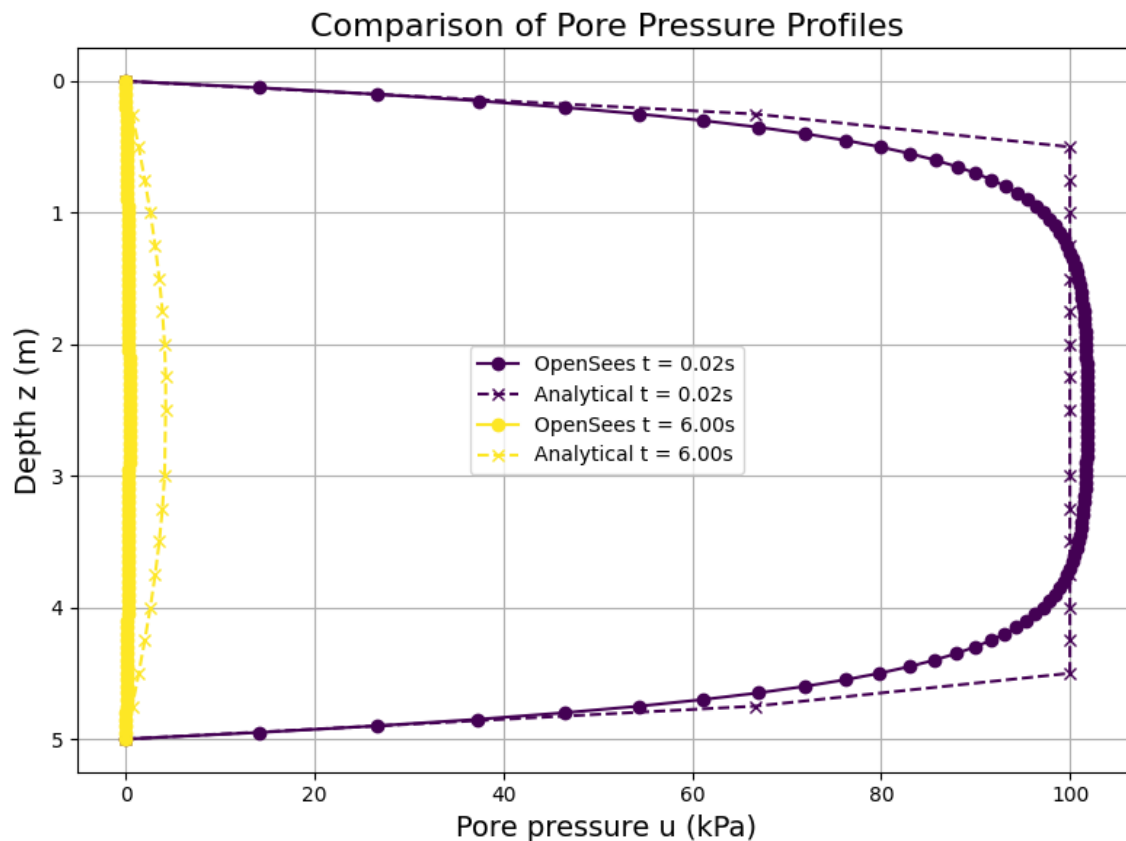
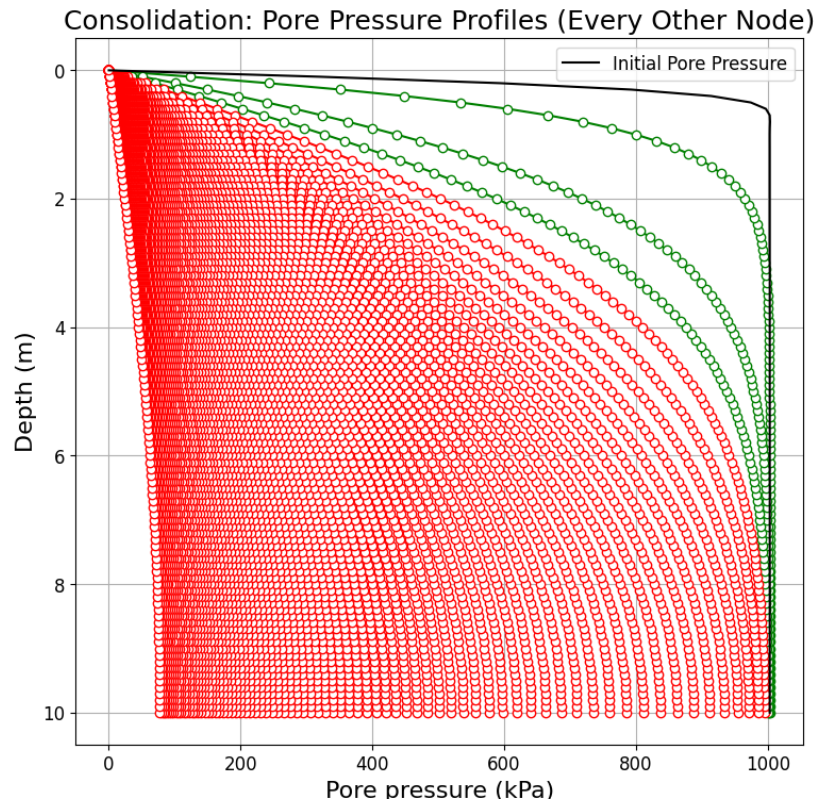


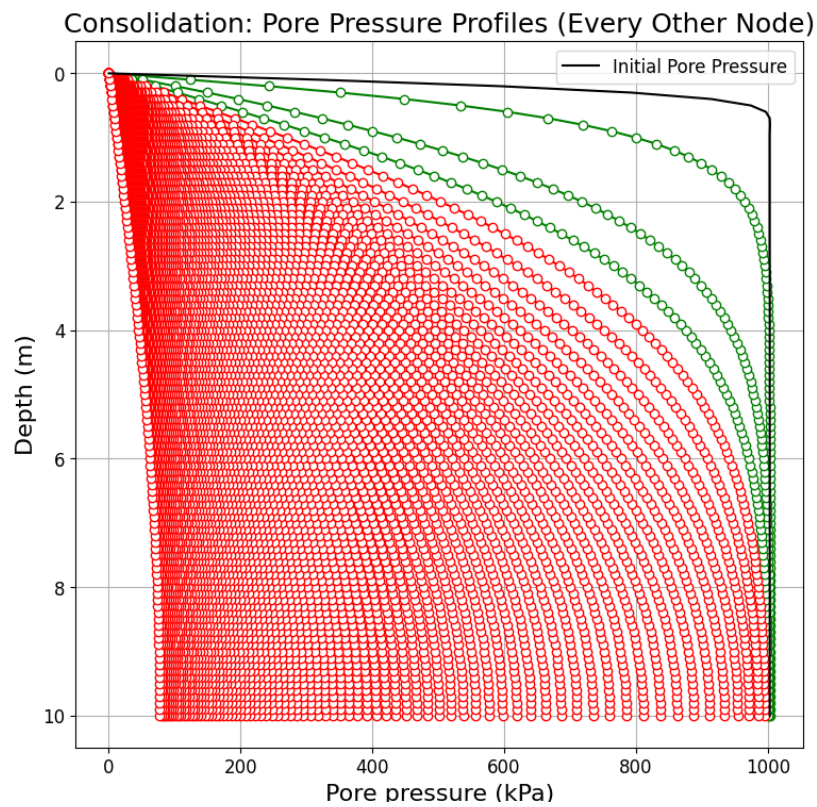
Fig. 5.13 Comparison between pore-water pressure profiles obtained using the New Element in OpenSees and the analytical–numerical finite difference solution at selected consolidation times.

The parameters adopted for the ManzariDafaliasIsoHard model are summarized in Table 5.2. The hydraulic properties and boundary conditions remained identical to those adopted in the linear elastic case, so that any differences in response are exclusively associated with the nonlinear constitutive behavior of the soil skeleton.

The results obtained with the ManzariDafaliasIsoHard model under fully saturated conditions are shown in Figures 5.14 and 5.15 for single- and double-drainage configurations, respectively.

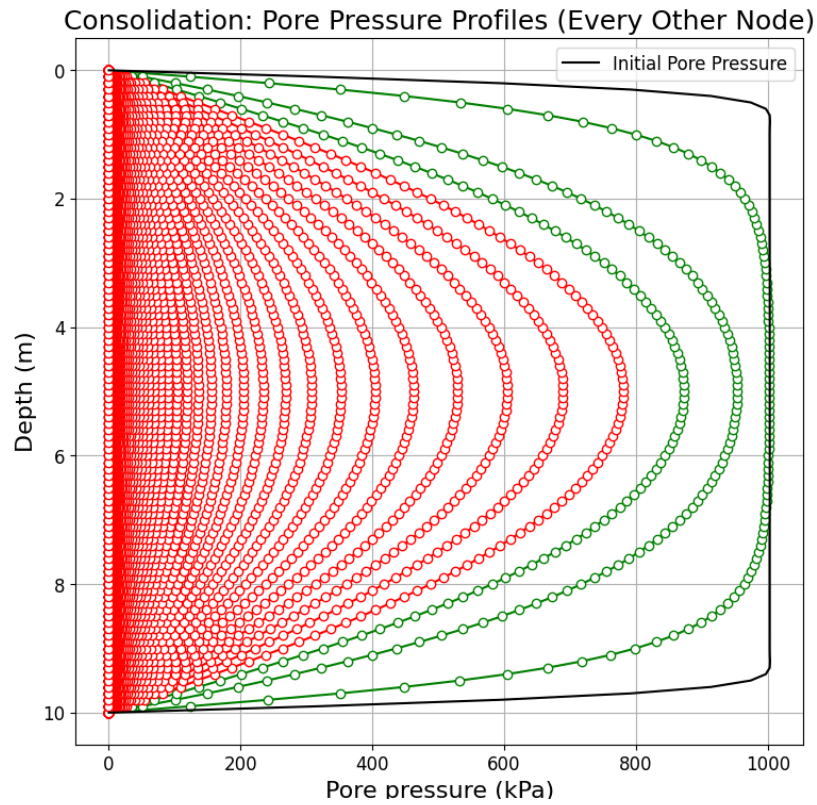


(a) QuadUP.

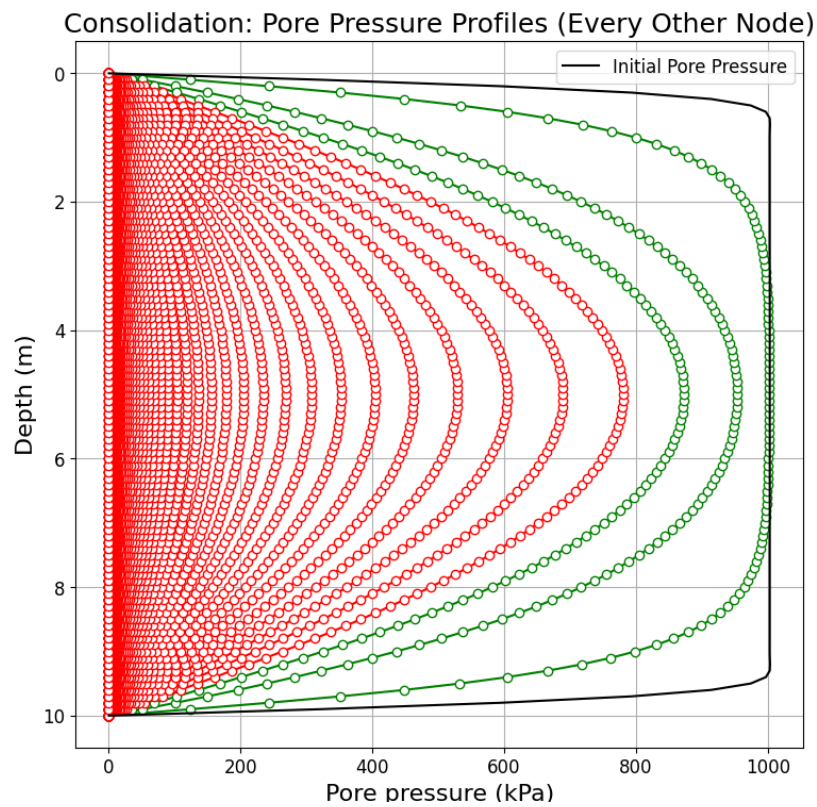


(b) New Element.

Fig. 5.14 Pore-water pressure profiles under single-drainage conditions for the layered soil configuration.



(a) QuadUP.



(b) New Element.

Fig. 5.15 Pore-water pressure profiles under single-drainage conditions for the layered soil configuration.

This agreement indicates that, for the fully saturated loading conditions considered, the response is mainly governed by the consolidation diffusion process. However, this should not be interpreted as a general independence from the constitutive behaviour of the soil skeleton, since the coefficient of consolidation depends on the soil compressibility m_v , or equivalently on E_{oed} . Thus, the observed agreement results from the similar compressibility mobilized by the two mechanical models in this specific benchmark.

The pore-water pressure dissipation profiles obtained with the elasto-plastic model are identical to those previously reproduced with the linear elastic material. Both the reference QuadUP formulation and the proposed New Element exhibit the same excess pore-water pressure magnitude, dissipation rate, and spatial distributions at the selected time instants.

Table 5.2 Parameters adopted for the ManzariDafaliasIsoHard model

Parameter	Symbol	Value	Unit
Shear modulus constant	G_0	549.59	kPa
Poisson's ratio	ν	0.30	—
Initial void ratio	e_{init}	0.65	—
Critical state stress ratio	M_c	1.25	—
Compression index	λ_c	0.022	—
Reference void ratio	e_0	0.80	—
State parameter constant	ξ	0.70	—
Atmospheric pressure	P_{atm}	101.3	kPa
Hardening parameter	h_0	7.0	—
Bounding surface parameters	c_h, n_b	0.90, 1.10	—
Fabric-dilatancy parameters	A_0, n_d	0.80, 1.00	—
Fabric evolution	z_{max}, c_z	4.0, 250.0	—
Mass density	ρ_s	1.7	t/m ³

5.4 Partially Saturated Column response under transient unsaturated conditions

As an additional verification case, a classical example of partially saturated flow was analyzed following the experimental and numerical study reported by Chan et al. [23]. This benchmark is particularly suitable for assessing the ability of the proposed formulation to capture the coupled hydraulic response under transient unsaturated conditions, including

the development of negative pore-water pressures, partial desaturation, and the associated reduction in hydraulic conductivity.

The emptying simulation was performed using the quadUPSVG element in a two-dimensional mesh reduced to one-dimensional response by enforcing equalDOF constraints (DOFs 1 and 2) between the left and right nodes at each elevation. The soil column ($H = 1.0\text{m}$) was discretized into 100 stacked quadrilateral elements. The mechanical behaviour of the soil skeleton was described using a linear elastic constitutive model, with the parameters specified in Table 5.1. Under unsaturated conditions, the hydraulic response was governed by the van Genuchten retention model, with the corresponding parameters and unsaturated hydraulic properties reported in Table 5.3. The numerical model and the boundary conditions adopted in the analysis are schematically shown in Figure 5.16.

Table 5.3 Van Genuchten parameters adopted in the proposed element.

α (kPa^{-1})	n (-)	m (-)	S_r^{res} (-)
0.065	Calculated from m via Eq. 2.5	0.66	0.001

The numerical simulation was carried out in two stages, corresponding to the initial gravity equilibrium stage and the subsequent transient emptying stage. In the first stage, shown in Figure 5.16a, the initial stress and pore-water pressure fields were generated under self-weight. Gravity was introduced directly at the element level by assigning a constant vertical body acceleration $g_y = 10 \text{ m/s}^2$ in the element command, rather than through an external load pattern. No external mechanical surcharge or imposed pore-pressure history was applied. The base of the column was fixed in both displacement degrees of freedom, while the pore-water pressure degree of freedom was fixed at the drainage boundaries. For the double-drainage configuration adopted in this analysis, zero pore-water pressure was imposed at both the top and bottom nodes during the initialization stage. Therefore, the initial condition corresponds to the equilibrium state obtained under self-weight with drained hydraulic boundaries at the top and bottom of the column.

After this initial equilibrium state was reached, the transient analysis was started by modifying the hydraulic boundary conditions, as illustrated in Figure 5.16b. Specifically, the pore-water pressure constraint at the top boundary was removed, while the bottom boundary remained drained with $p_w = 0$. As a result, the upper boundary became a natural no-flow boundary in the u - p_w formulation, whereas the bottom boundary acted as the active drainage boundary during the subsequent transient response. This change in hydraulic boundary conditions initiated the emptying process and allowed water to flow out through the base of the column.

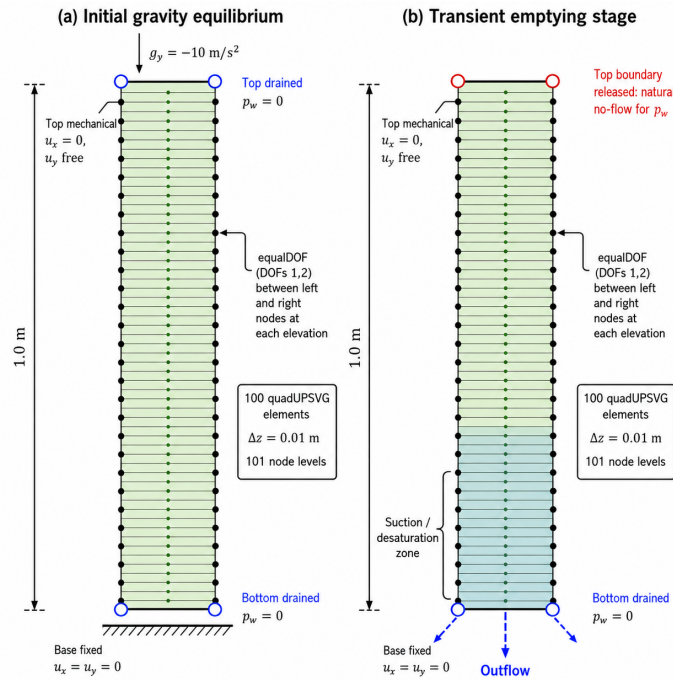


Fig. 5.16 Numerical model and boundary conditions adopted in the analysis.

During the transient stage, the evolution of pore-water pressure, suction, degree of saturation S_r , hydraulic conductivity, and Darcy flux was monitored. As the pore-water pressure decreased near the drainage boundary, the van Genuchten retention law controlled the progressive transition from saturated to partially saturated conditions. Consequently, negative pore-water pressures developed consistently with the soil-water retention curve, while the hydraulic conductivity decreased as the degree of saturation was reduced.

The results of this preliminary analysis, obtained using the New Element, are shown in Figures 5.17–5.20. Figure 5.17 presents the evolution of the pore-water pressure profiles during the emptying process, highlighting the development of negative pressures near the drainage boundary and their progressive propagation into the soil column. The corresponding effective stress response is illustrated in Figure 5.18, showing a redistribution of effective stresses associated with the reduction in pore-water pressure. Figures 5.19 and 5.20 depict the evolution of the degree of saturation and the hydraulic permeability, respectively, confirming the formation of a partially saturated zone characterized by reduced saturation and strongly diminished permeability.

The same analysis was also repeated using the proposed quadUPSVG element combined with the ManzariDafaliasIsoHard constitutive model. The corresponding mechanical parameters are reported in Table 5.2, while the hydraulic response was kept unchanged and

governed by the van Genuchten relationship summarized in Table 5.3. The corresponding results are presented in Figures 5.21–5.24.

The response obtained with the `ManzariDafaliasIsoHard` model coincides with that of the linear elastic formulation, as the stress paths developed during the transient process remain within the elastic domain of the constitutive model. Since the yield surface is not reached, no plastic straining is activated. This outcome is expected, as the present benchmark is governed primarily by seepage-driven hydraulic transients rather than by significant mechanical loading. Consequently, for the subsequent flow-dominated examples presented in this chapter, the soil skeleton is modeled using a linear elastic constitutive description.

While the emptying analysis provides valuable insights into the qualitative behavior of the formulation under unsaturated conditions, a more rigorous verification was subsequently carried out by adopting a staged numerical procedure consistent with the framework proposed by Zienkiewicz et al. [176] and further documented by Chan et al. [23]. The analysis was performed on a two-dimensional domain of width $W = 1\text{m}$ and height $H = 10\text{m}$, discretized using a structured mesh of 40×20 quadrilateral `quadUPVSG` elements. The elastic properties of the soil skeleton adopted in this analysis correspond to those previously summarized in Table 5.1, while the parameters governing the soil–water retention curve are specified in Table 5.3. The staged procedure involves a sequence of loading and boundary condition phases specifically designed to isolate and verify the coupled hydro-mechanical response of the formulation.

In the present `OpenSees` implementation, the hydraulic regime is controlled through the pore-water pressure degree of freedom of the coupled u - p_w finite element. Since the formulation adopts the solid displacement field and the pore-water pressure as primary variables, the water flux is not introduced as an independent unknown, but is obtained from the pressure gradient through Darcy’s law. Therefore, drained boundaries are prescribed by imposing a pore-water pressure value, typically $p_w = 0$, whereas boundaries where no pore-pressure condition is imposed behave as natural no-flow boundaries.

Within this framework, the hydraulic boundary conditions were applied through a staged procedure, which allowed the drainage path and the transient flow response to be controlled consistently with both the finite element formulation and the `OpenSees` implementation. This staged strategy was necessary because, in order to generate a hydraulic gradient in a given direction, the corresponding pore-water pressure constraints must be removed from the boundaries through which the pressure field is allowed to evolve. At the same time, when lateral mechanical constraints are released to avoid artificial confinement during the subsequent stage, the equilibrium state reached at the end of the previous stage must be preserved. For this reason, in `OpenSees`, the reactions computed at the constrained

boundaries at the end of the preceding stage were replaced by equivalent mechanical nodal forces before releasing the corresponding constraints. This procedure made it possible to modify the hydraulic boundary conditions without introducing spurious changes in the mechanical equilibrium of the soil column. The same methodology was adopted in all the numerical examples in which different flow boundary conditions were considered.

In the first stage (Stage A), the hydrostatic initial condition was generated under self-weight. This stage was used only to establish the initial pore-water pressure field, corresponding to the hydrostatic distribution $\gamma_w h$, before applying the hydraulic boundary conditions required for the subsequent transient analysis. In this stage, the appropriate pore-water pressure constraints were imposed to define the initial hydraulic state, while the mechanical equilibrium of the soil column was reached under gravity loading. The resulting hydrostatic pore-water pressure distribution obtained at the end of Stage A is shown in Figure 5.25.

From this point onward, all numerical analyses were initialized through this Stage A procedure, in which the hydrostatic pore-water pressure condition was first established before modifying the hydraulic boundary conditions according to the drainage configuration considered.

The evolution of the degree of saturation during Stage A is presented in Figure 5.28. The results indicate that the soil column remains fully saturated throughout this stage. This response is consistent with the pore-water pressure observed in Figure 5.25 and confirms that the hydraulic response is correctly captured.

In the subsequent stage (Stage B), the top drainage condition was removed and equivalent mechanical loads were applied along the lateral boundaries to preserve equilibrium while lateral constraints were released. The pore-water pressure snapshots corresponding to this stage are also shown in Figure 5.26. The associated saturation fields in Figure 5.29 confirm that the soil remains close to full saturation during this stage, indicating that the mechanical adjustment does not introduce spurious hydraulic effects.

Finally, in Stage C, a full basal drainage condition was imposed, allowing pore-water pressures to dissipate through the base of the column. The resulting pore-water pressure fields at selected times are shown in Figure 5.27. A clear downward propagation of the drainage front is observed, with pore-water pressure progressively decreasing from the base upward. This behavior is consistent with the imposed drainage boundary and the expected transient response of an unsaturated column subjected to basal drainage.

The corresponding evolution of the degree of saturation during Stage C is presented in Figure 5.30. In contrast to Stages A and B, a well-defined unsaturated zone develops near the base of the column, with a gradual reduction in saturation propagating upward over time.

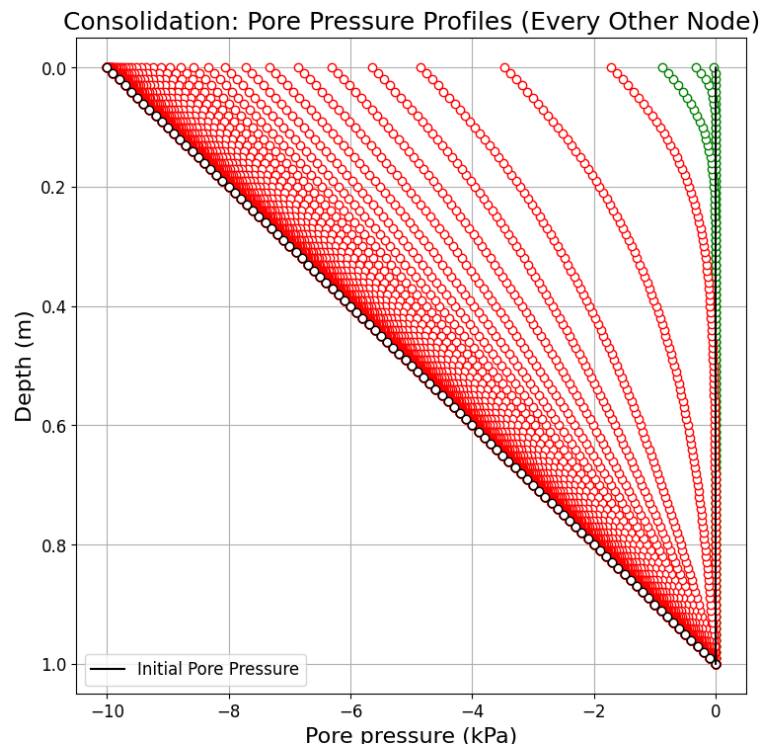


Fig. 5.17 New Element Pore-Water Pressures

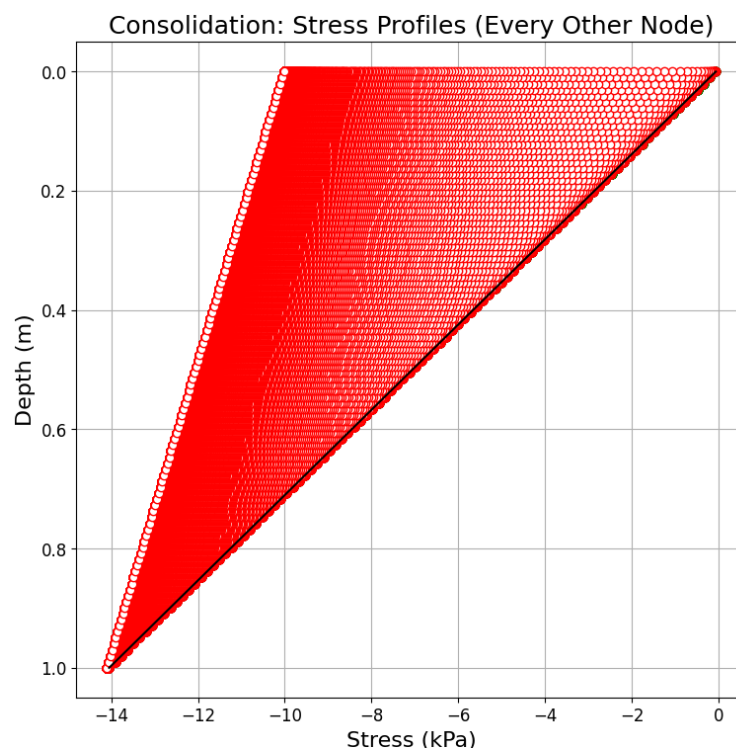


Fig. 5.18 New Element Effective Stresses

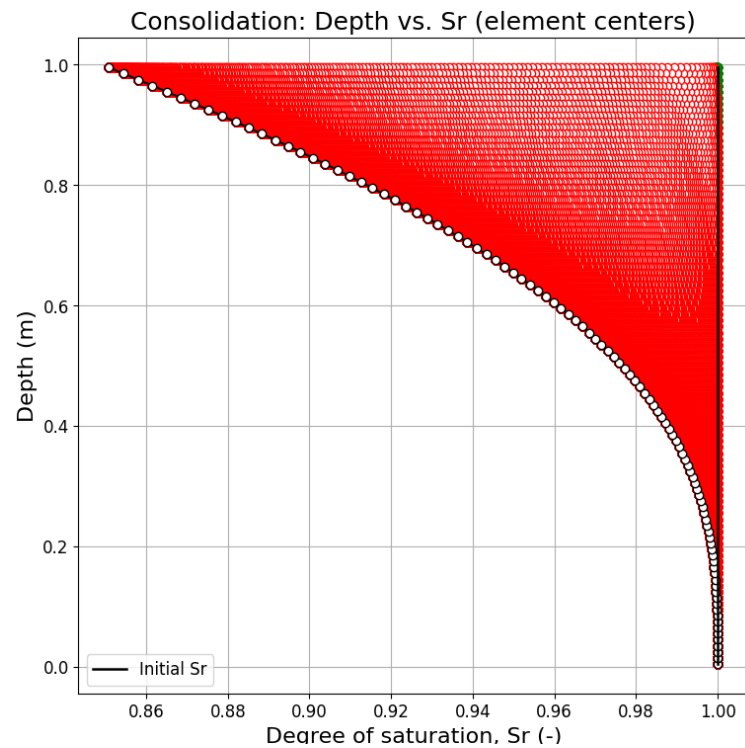


Fig. 5.19 New Element Degree of Saturation

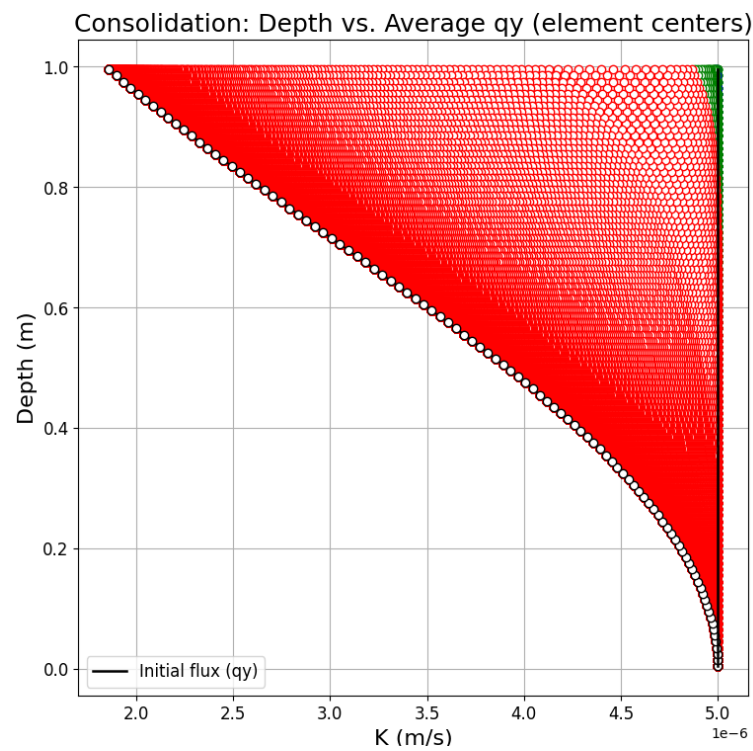


Fig. 5.20 New Element Permeability

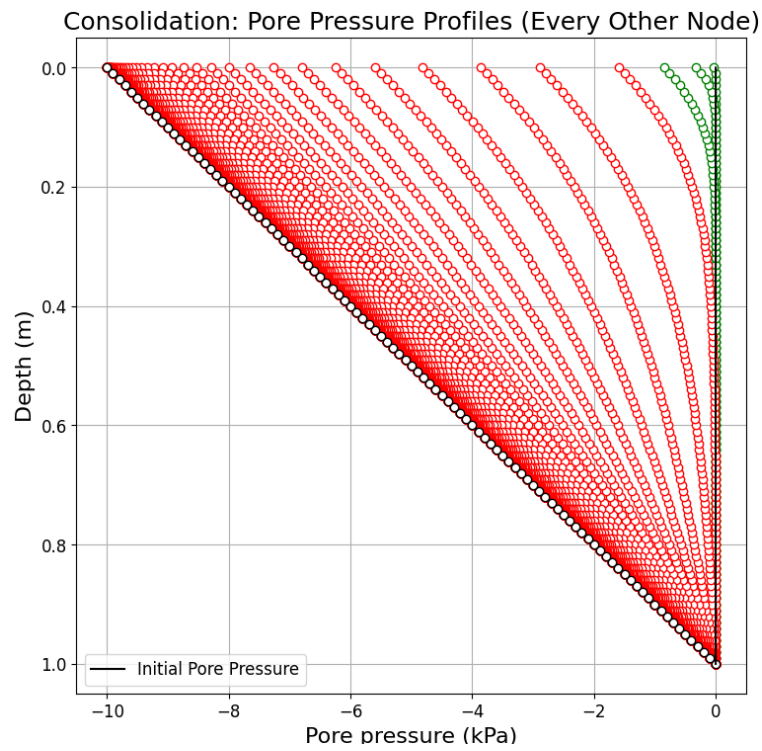


Fig. 5.21 New Element Pore-Water Pressures (Manzari Dafalias IsoHard)

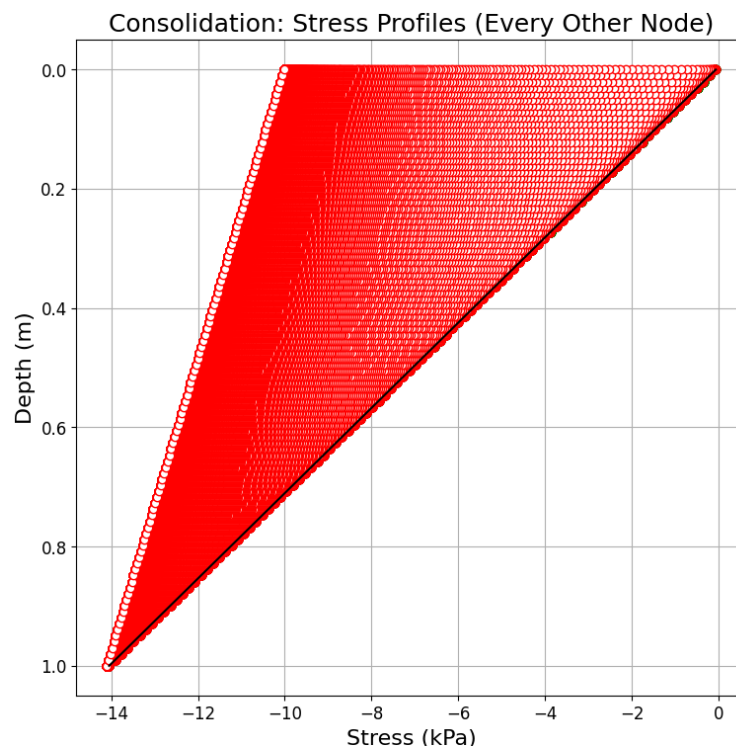


Fig. 5.22 New Element Effective Stresses (Manzari Dafalias IsoHard)

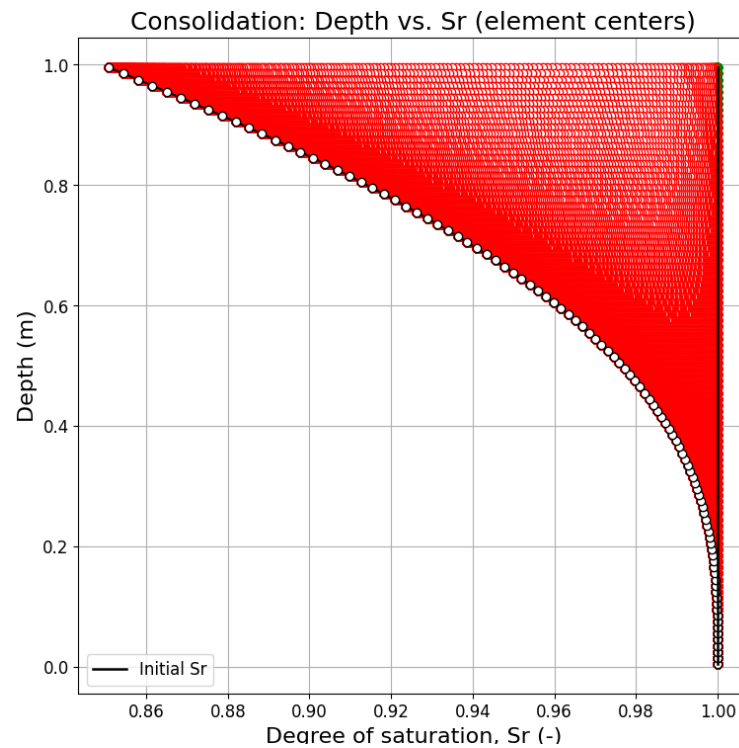


Fig. 5.23 New Element Degree of Saturation (Manzari Dafalias IsoHard)

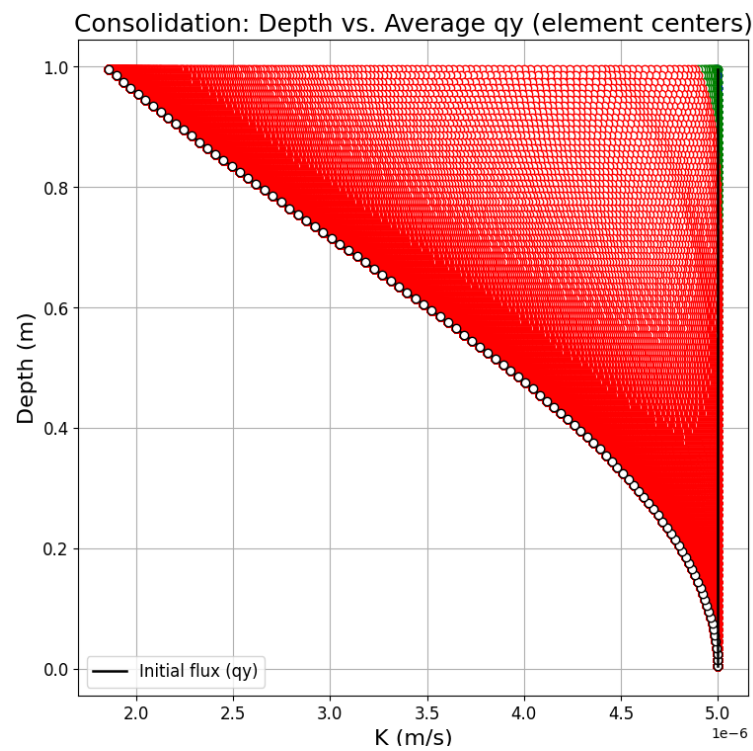


Fig. 5.24 New Element Permeability (Manzari Dafalias IsoHard)

Stage A — pressure field snapshots

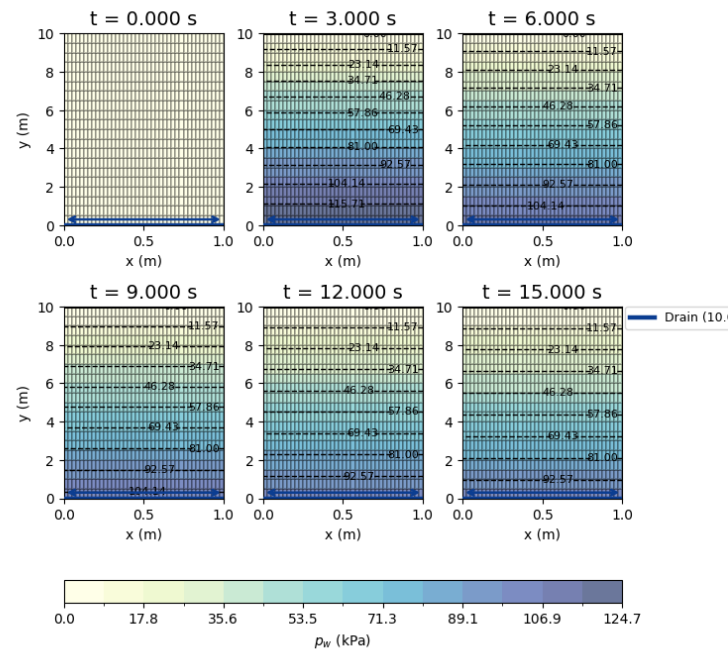


Fig. 5.25 Pore-water pressure snapshots at Stages A.

Stage B — pressure field snapshots

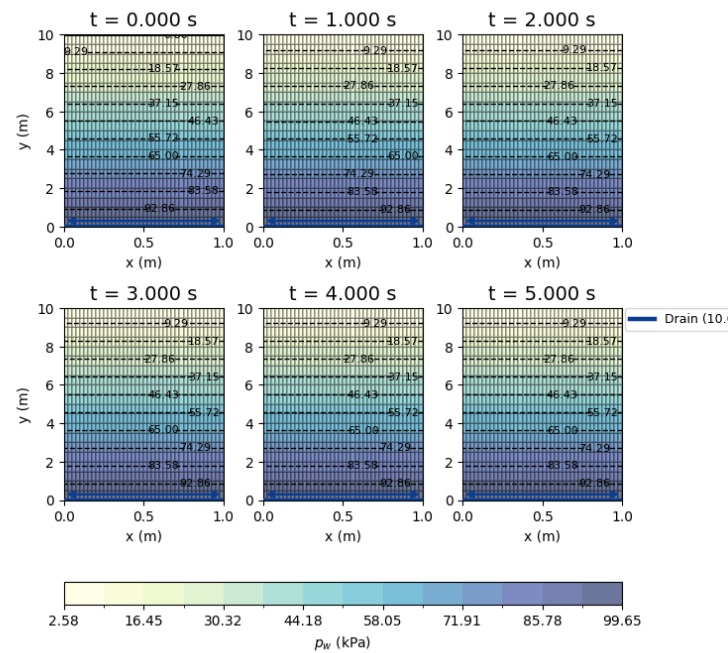


Fig. 5.26 Pore-water pressure snapshots at Stages B.

Stage C — pressure field snapshots

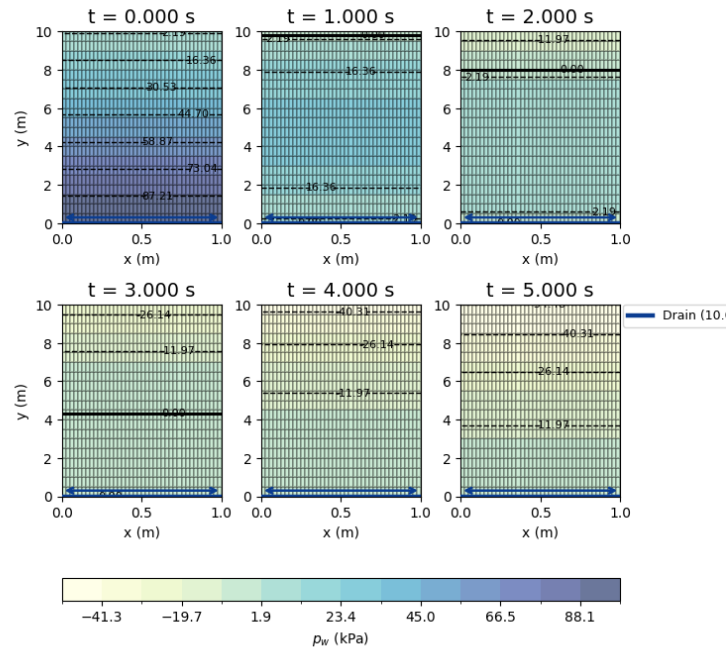
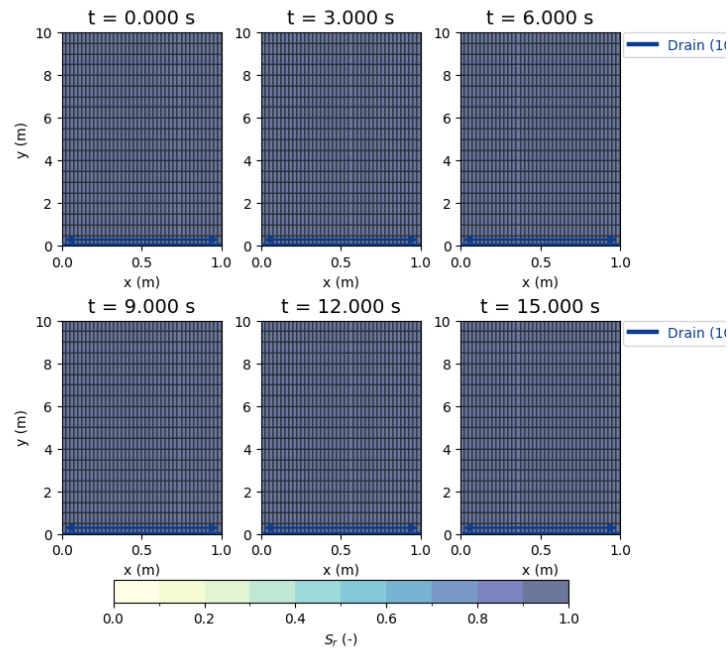
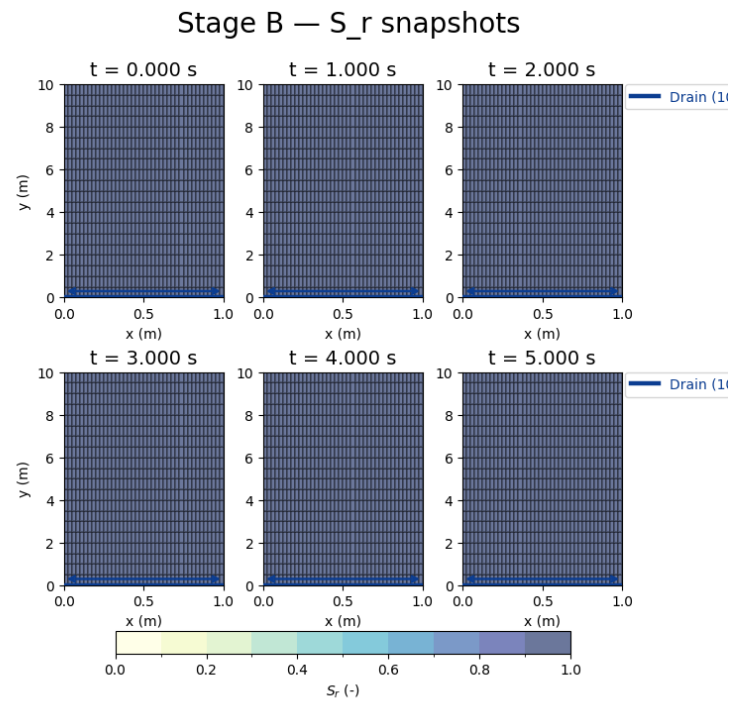
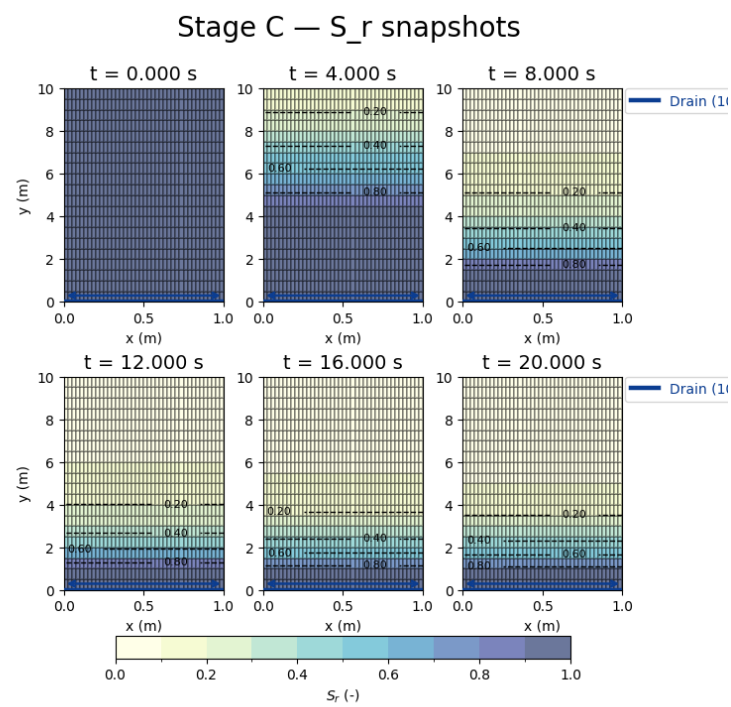


Fig. 5.27 Pore-water pressure snapshots at Stage C.

Stage A — S_r snapshotsFig. 5.28 S_r snapshots at Stages A.

Fig. 5.29 S_r snapshots at Stages B.Fig. 5.30 S_r snapshots at Stage C.

The spatial correlation between the regions of reduced pore-water pressure in Figure 5.27 and the desaturation patterns in Figure 5.30 demonstrates a coherent coupling between suction development and saturation evolution within the proposed formulation.

The staged analysis framework provides a consistent basis for comparing the proposed formulation against reference results reported in the literature. Simulations performed with the New Element show trends in pore-water pressure dissipation and saturation evolution that agree well with those reported by Chan et al. [23]. This agreement confirms that the element correctly captures the essential coupling between pore-water pressure and degree of saturation governing partially saturated flow.

Overall, this example demonstrates that the proposed finite element formulation is capable of reproducing both the qualitative features observed during the transient emptying of a soil column and the more rigorous benchmark behavior associated with staged unsaturated flow analyses. These results provide further verification of the correctness and robustness of the implementation and establish confidence in its application to more complex partially saturated geotechnical boundary value problems.

5.5 Refilling (Re-saturation) of a Partially Drained Soil Column

To complement the previously analyzed emptying scenario, an additional stage (Stage D) was incorporated into the original staged procedure to investigate the refilling (re-saturation) behaviour of the soil column. While the initial benchmark focused exclusively on drainage-induced desaturation (Stages A–C), this extended configuration examines the reverse hydraulic process, in which the phreatic level is restored from the top boundary. The same geometry, discretization, mechanical properties (Table 5.1), and soil–water retention parameters (Table 5.3) were adopted. The analysis was performed using both the reference QuadUP element and the proposed quadUPSVG element, allowing a direct comparison of their responses under identical hydraulic boundary conditions.

Starting from the final state of Stage C, a full hydraulic boundary condition ($p_w = 0$) was imposed along the entire top boundary using a MultipleSupport pattern, while the lateral and basal boundaries were left under natural no-flow conditions. This additional stage completes the hydraulic cycle by reversing the previously imposed drainage condition.

Stage C — pressure field snapshots

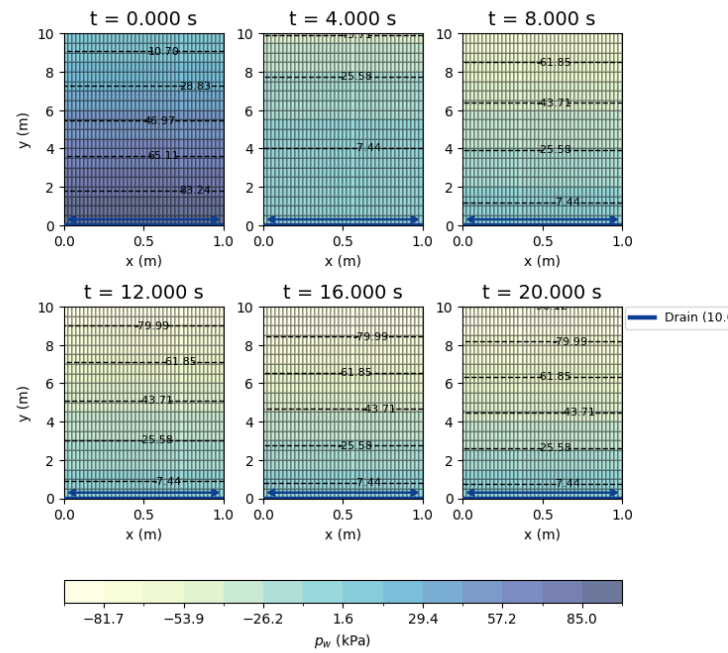


Fig. 5.31 Pore-water pressure snapshots at Stage C QuadUP.

Stage C — pressure field snapshots

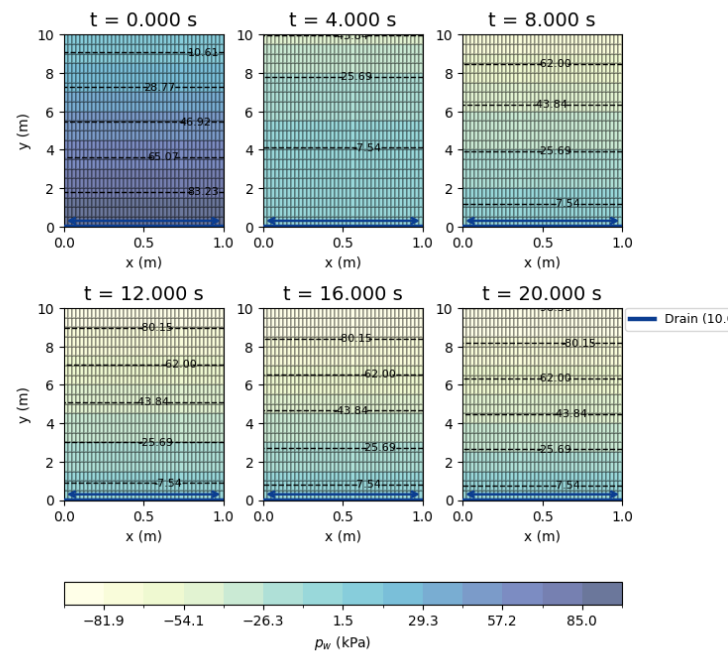


Fig. 5.32 Pore-water pressure snapshots at Stage C New element.

Stage D — pressure field snapshots

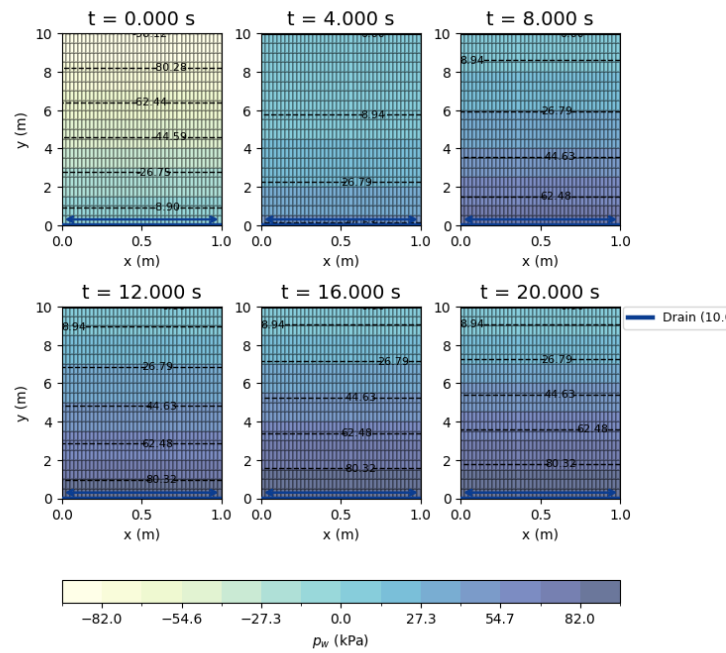


Fig. 5.33 Pore-water pressure snapshots at Stage D QuadUP.

Stage D — pressure field snapshots

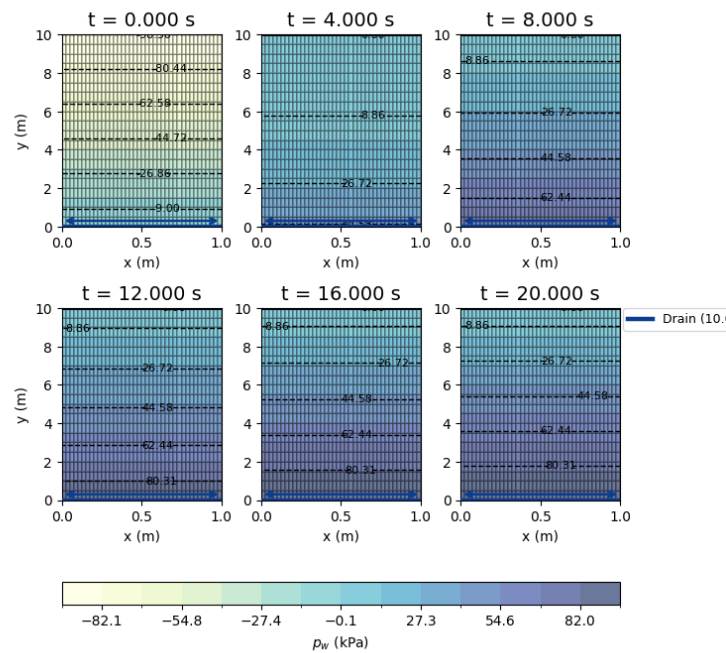


Fig. 5.34 Pore-water pressure snapshots at Stage D New element.

Stage C — pressure field snapshots

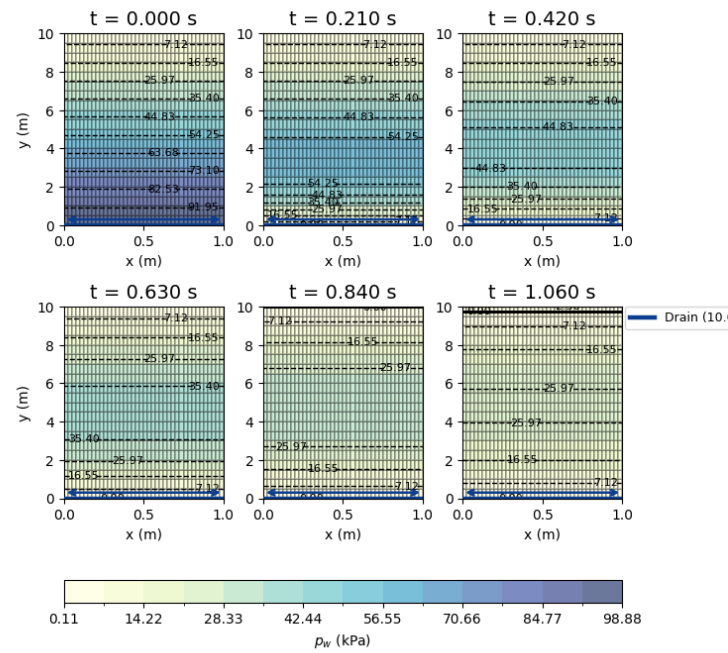


Fig. 5.35 Pore-water pressure snapshots at Stage C New element.

Stage D — pressure field snapshots

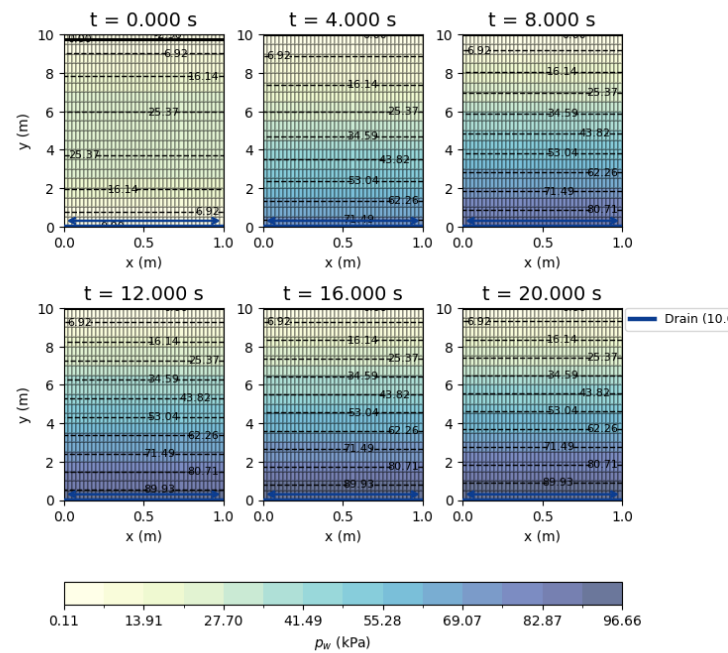


Fig. 5.36 Pore-water pressure snapshots at Stage D New element.

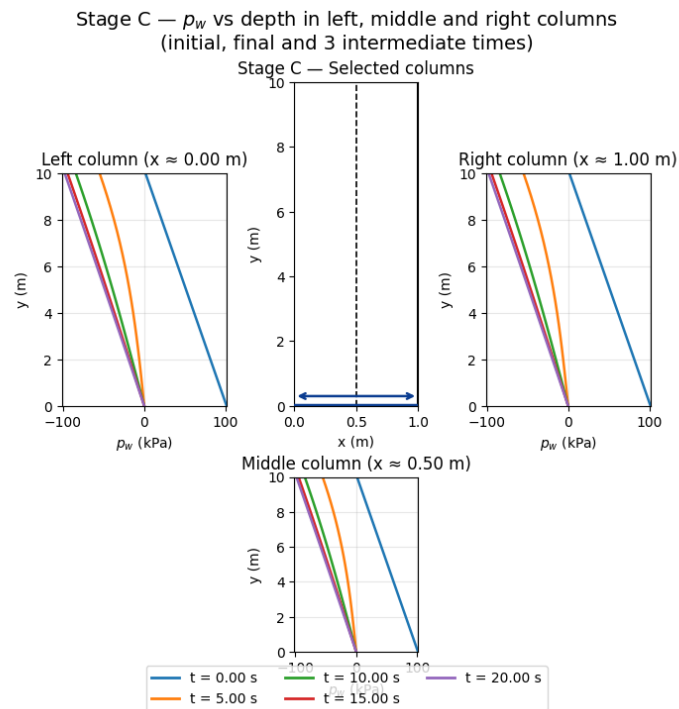


Fig. 5.37 Pore-water pressure snapshots at Stage C QuadUP.

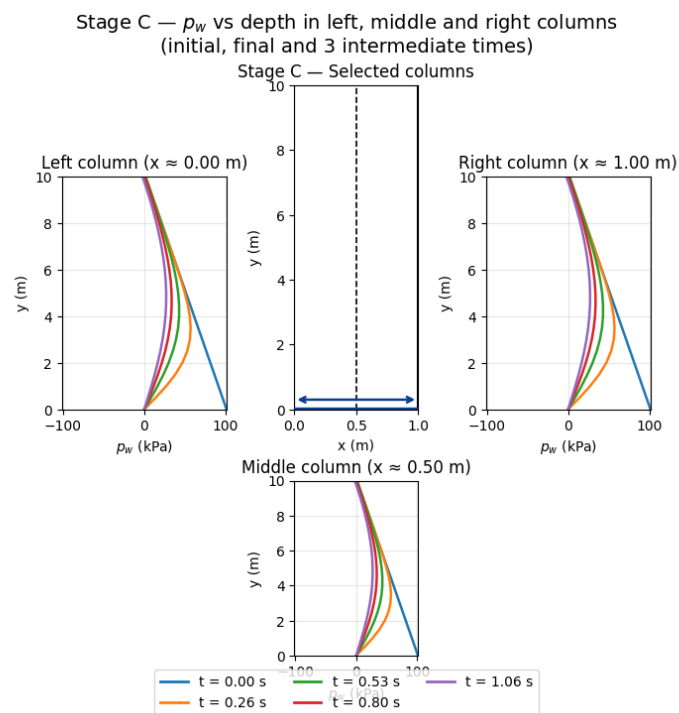


Fig. 5.38 Pore-water pressure snapshots at Stage C New Element.

The pore-water pressure distributions corresponding to the drainage phase (Stage C) are shown in Figures 5.31 and 5.32 for the QuadUP and quadUPSVG elements, respectively. The subsequent refilling response (Stage D) is presented in Figures 5.33 and 5.34. A different dissipation behaviour can be observed between the two element formulations. While the reference QuadUP element reproduces the classical saturated hydraulic response, the proposed quadUPSVG formulation exhibits a moderately altered pressure evolution governed by the soil–water retention relationship. In the latter case, both the rate and spatial distribution of dissipation are influenced by the retention curve parameters, allowing direct control of the coupled suction–saturation response.

In this sense, the hydraulic behaviour was calibrated to represent a clean sand using the van Genuchten parameters $\alpha = 0.0816 \text{ kPa}^{-1}$, $m = 0.083$, and $S_r^{\text{res}} = 0.179$. These values define the suction–saturation coupling that regulates the transient response during drainage. The influence of this calibration is clearly observed in Figures 5.37 and 5.38, which compare the Stage C pressure fields obtained with the QuadUP and quadUPSVG formulations.

While the QuadUP element exhibits the classical diffusion-type dissipation associated with a purely saturated formulation, the quadUPSVG element shows a modified pressure redistribution governed by suction development and the retention-controlled permeability evolution. In particular, the limited negative pore-water pressures observed near the drainage boundary in the quadUPSVG results are physically consistent with the expected behaviour of a clean sand, where suction develops but remains moderate due to the relatively low air-entry value.

These differences confirm that the Stage C response is significantly influenced by the retention model and is more realistically represented by the proposed unsaturated formulation.

5.6 Seepage in a confined rectangular domain

A two-dimensional seepage problem was simulated to verify the correct implementation of hydraulic and mechanical boundary conditions in the New Element formulation. The soil was modeled as a clean sand, and the analyses were performed using both the newly implemented element and the reference QuadUP element, allowing direct comparison of their performance.

The numerical model consists of a confined rectangular sand domain discretized using a structured mesh of four-node quadrilateral elements, as shown in Figure 5.39. The domain was discretized using 40×20 elements with uniform dimensions of 0.5 m, resulting in a total of 800 elements. A drainage line of 4 m was imposed along the bottom-right portion of the base to induce seepage and partial desaturation within the soil domain.

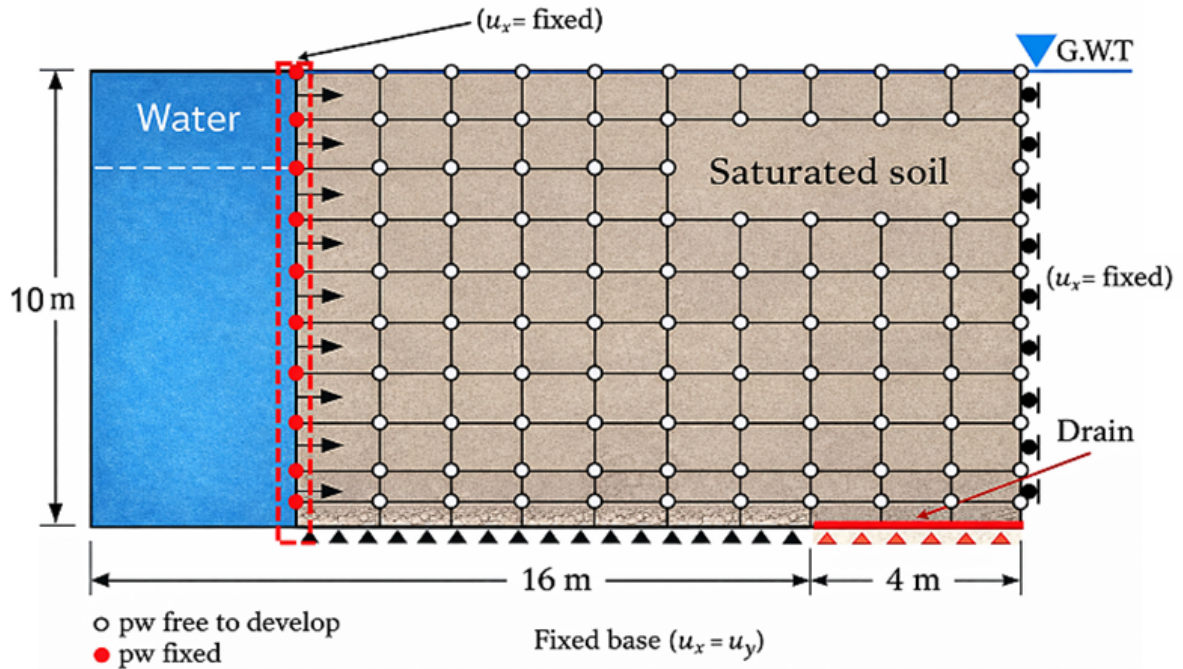


Fig. 5.39 Numerical model of the hydrostatic bucket test with localized base drainage.

Although the classical QuadUP formulation may develop negative pore-water pressures to satisfy equilibrium under transient conditions, these pressures are computed within a fully saturated framework and are not constrained by a soil-water retention relationship. As a result, suction levels may be overestimated and are not consistently linked to variations in degree of saturation. In contrast, the proposed formulation incorporates a retention law that governs the evolution of matric suction and saturation, providing a physically consistent description of partially saturated conditions. This benchmark therefore allows highlighting the differences between a purely saturated response and a retention-controlled unsaturated behavior.

For the hydraulic response of the clean sand, the same van Genuchten parameters adopted in the previous example were employed, namely $\alpha = 0.0816 \text{ kPa}^{-1}$, $m = 0.083$, and $S_r^{\text{res}} = 0.179$, together with the intrinsic saturated permeability reported in Table 5.1. These parameters govern the suction–saturation relationship and the permeability degradation within the partially saturated zone. Three stages were analyzed:

- **Stage A:** The top boundary was drained ($p_w = 0$), while the base and vertical sides were hydraulically impermeable. Mechanically, the base was fully fixed ($u_x = u_y = 0$), and the lateral sides were constrained in the horizontal direction ($u_x = 0$), but free vertically. This stage established initial pore-water pressure conditions under gravity loading.

- **Stage B:** The top drainage was sealed, and the reactions from Stage A were replaced by equivalent lateral loads on the vertical boundaries. Subsequently, the lateral fixities in u_x were released, leaving the system in a state of hydro-mechanical equilibrium without lateral restraints.
- **Stage C:** A localized drain was introduced along the bottom-right edge (last 4 m of the base) by prescribing zero pore-water pressure at those nodes. Simultaneously, the pore-water pressures computed on the left boundary at the end of Stage B were imposed as constant values throughout this stage. This configuration created a controlled flow path, enabling the assessment of pore-pressure dissipation and hydraulic fluxes under confined conditions.

This benchmark provides a comprehensive verification regarding the effectiveness of both displacement and pore-pressure boundary conditions, ensuring that the element correctly reproduces seepage, confinement, and drainage mechanisms within a rectangular soil domain. The results obtained with the New Element and the reference QuadUP element are presented in the following section. Figures 5.40 and 5.41 show the combined pore-water pressure maps and sectional profiles obtained at the end of Stage A ($t = 0$ s) using the New Element and the QuadUP element, respectively. At this stage, the top boundary is fully drained, while the remaining boundaries are hydraulically impermeable, leading to a hydrostatic pore-pressure distribution governed by gravity loading.

Both figures highlight a linear increase of pore-water pressure with depth, as expected for a confined saturated soil under self-weight. The sectional profiles extracted at Sections A–A, B–B, and C–C are nearly identical for the two formulations, confirming that the initial hydraulic state is consistently reproduced by both elements. The pore-water pressure increases from approximately $p_w = 0$ kPa at the top boundary to $p_w = 100$ kPa at the base, with no differences between the two elements.

The corresponding vertical stress distributions at the end of Stage A are shown in Figures 5.44 and 5.45 for the New Element and QuadUP, respectively. The stress fields display a smooth increase with depth, reflecting the contribution of self-weight and the absence of seepage-induced stress perturbations. From the sectional profiles, the effective vertical stress increases linearly from approximately $\sigma'_{yy} = 0$ kPa at the surface to about $\sigma'_{yy} \approx 65\text{--}70$ kPa at the base. The results are the same between the two elements at this stage and verify the correct implementation of gravity loading and mechanical boundary conditions.

The transition from Stage A to Stage B involves the removal of the top drainage condition and the release of lateral horizontal constraints, while maintaining overall equilibrium

through equivalent boundary loads. Although no drainage is allowed during this stage, this configuration tests the consistency of the hydro-mechanical coupling under changing mechanical boundary conditions.

The pore-water pressure distributions at the end of Stage B remain essentially unchanged with respect to Stage A, indicating that the imposed boundary conditions preserve the hydraulic equilibrium. This behavior is consistently captured by both the New Element and the QuadUP formulation, demonstrating the stability of the numerical implementation under constraint reconfiguration.

The introduction of a localized drain along the bottom-right portion of the base during Stage C induces a transient seepage process under confined conditions. Figures 5.42 and 5.43 present the combined pore-water pressure maps and sectional profiles obtained at $t = 5$ s using the New Element and $t = 10$ s the QuadUP element, respectively.

Both formulations capture a pronounced pore-pressure gradient directed toward the localized drain, with a progressive reduction in pore-water pressure in the vicinity of the drainage boundary. The sectional profiles reveal consistent pressure dissipation along the flow path, while regions farther from the drain retain higher pore-water pressures, as expected under confined seepage conditions.

A closer inspection of the sectional profiles at $t = 5$ s and 10 s shows that near Section B–B (adjacent to the drain) the QuadUP formulation predicts pore-water pressures reaching approximately $p_w \approx -40$ to -50 kPa, reflecting a purely saturated diffusion-controlled response. In contrast, the New Element exhibits a moderated pressure reduction governed by the van Genuchten retention relationship, resulting in a smoother pressure gradient and a physically consistent suction development for a clean sand. The spatial extent of negative pore pressures is therefore more controlled in the proposed formulation.

The corresponding evolution of vertical stresses during Stage C is illustrated in Figures 5.46 and 5.47, showing results at $t = 5$ s and $t = 10$ s, respectively. As pore-water pressure dissipates near the drainage boundary, an increase in effective vertical stress is observed in the same region, reflecting the transfer of load from the pore fluid to the soil skeleton.

At $t = 5$ s (Fig. 5.46), the New Element predicts a localized stress increase near Section B–B reaching approximately $\sigma'_{yy} \approx 150$ – 165 kPa, while regions farther from the drain remain close to the initial hydrostatic stress distribution.

At $t = 10$ s (Fig. 5.47), the stress concentration becomes more pronounced. The effective vertical stress near the drainage boundary reaches approximately $\sigma'_{yy} \approx 160$ – 170 kPa. While both formulations capture this increase, the New Element exhibits a slightly more localized

redistribution consistent with the retention-controlled permeability reduction and suction coupling.

These quantitative observations confirm that both elements correctly reproduce the confined seepage mechanism and the associated effective stress transfer. However, the New Element provides a more physically consistent representation of partially saturated behavior near the drainage boundary, particularly in the prediction of suction development, permeability degradation, and localized stress redistribution.

The spatial and temporal evolution of the degree of saturation is illustrated in Figures 5.48–5.50 for Stages A, B, and C, respectively, using the New Element. During Stage A (Fig. 5.48), the soil domain remains fully saturated, as expected, since drainage is only permitted at the top boundary and no seepage path is present. This confirms that the initial hydrostatic condition is correctly preserved.

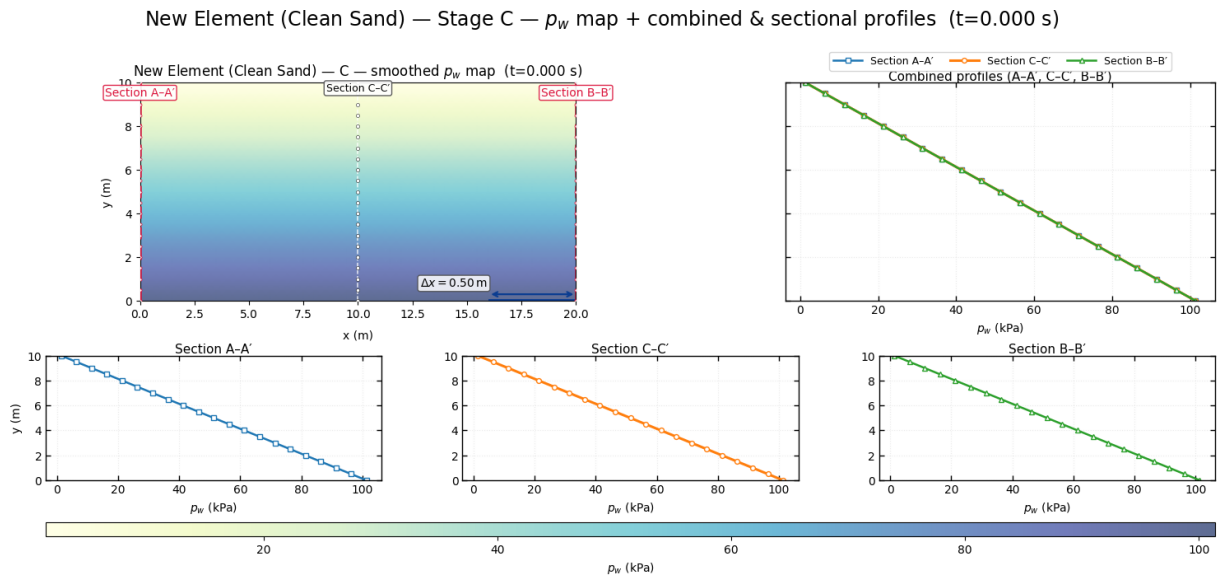
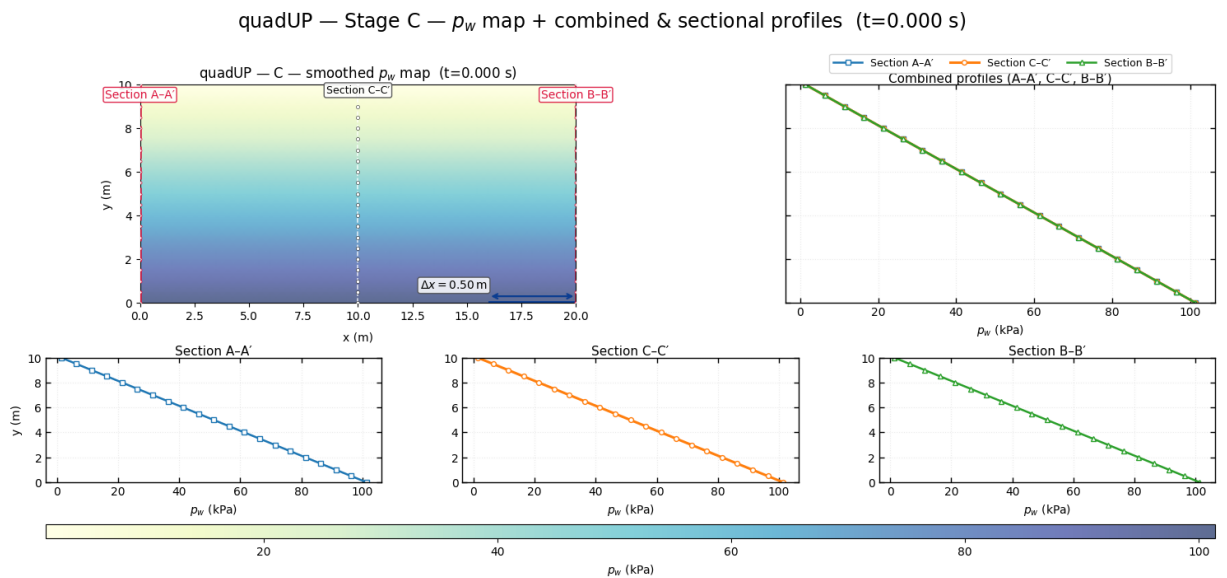
A fully saturated state is also observed during Stage B (Fig. 5.49), despite the modification of mechanical boundary conditions, indicating that the reconfiguration of constraints does not induce spurious hydraulic effects.

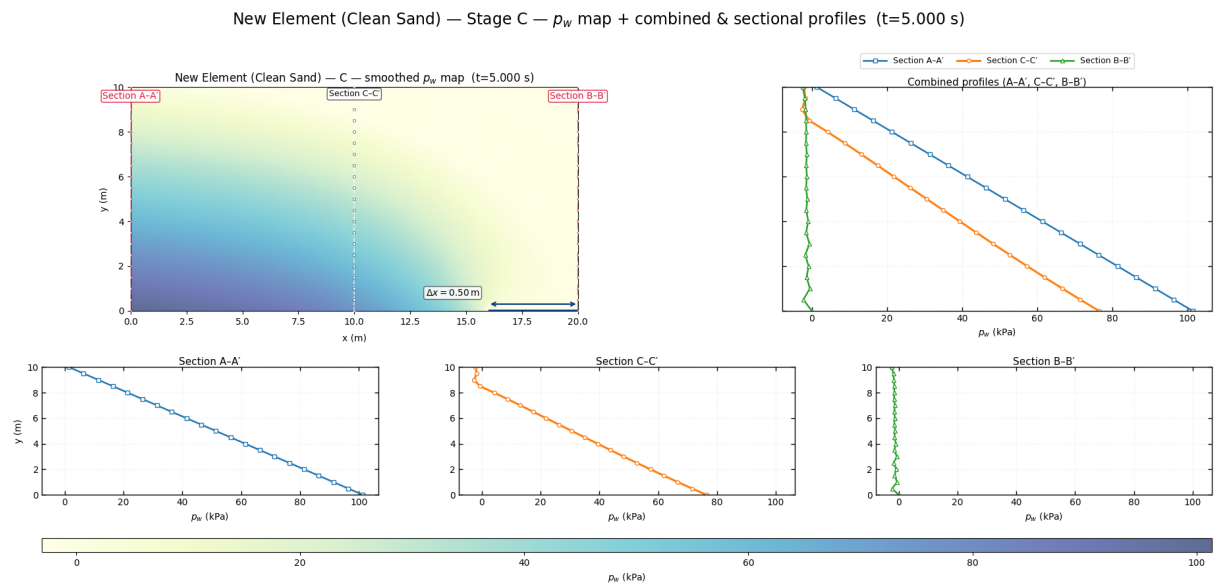
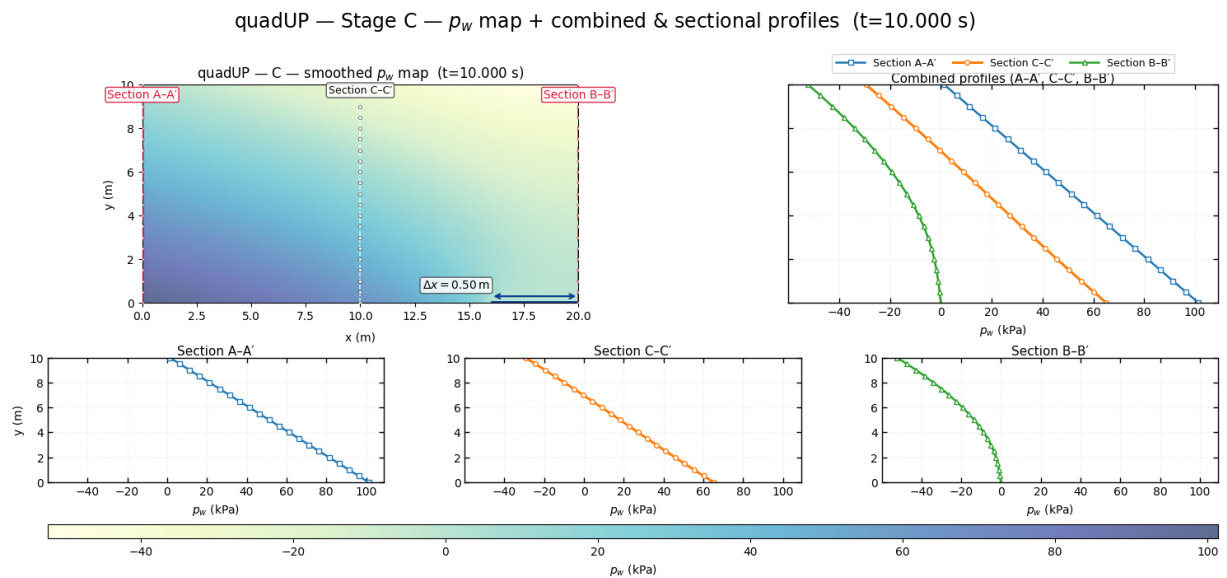
In contrast, Stage C (Fig. 5.50) is characterized by the progressive development of a partially saturated zone near the localized drain. A distinct desaturation front propagates into the soil domain following the imposed seepage path, demonstrating that the formulation correctly captures the coupling between pore-water pressure reduction and saturation evolution.

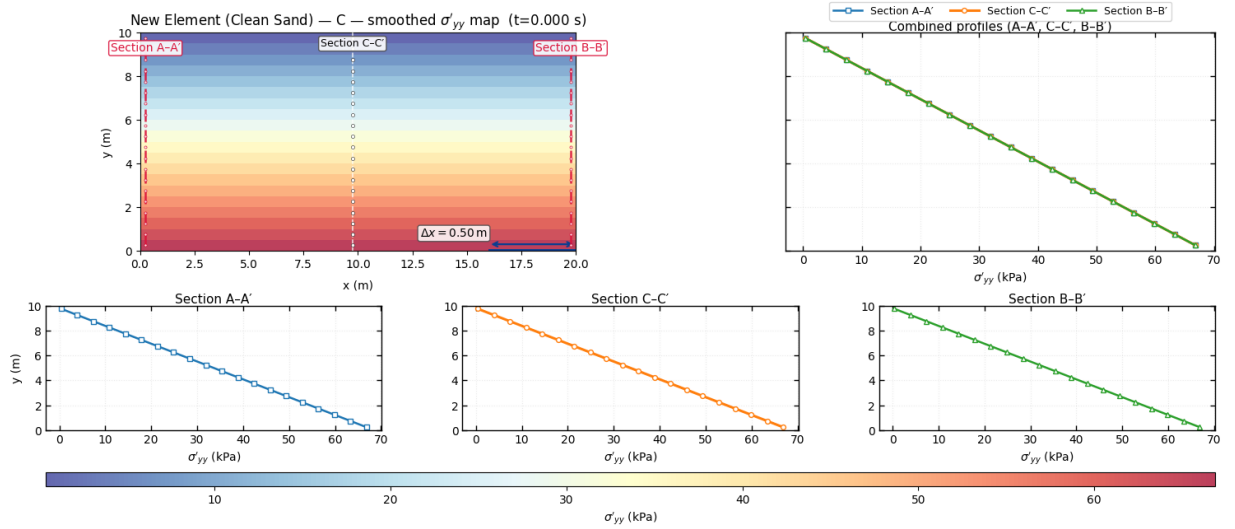
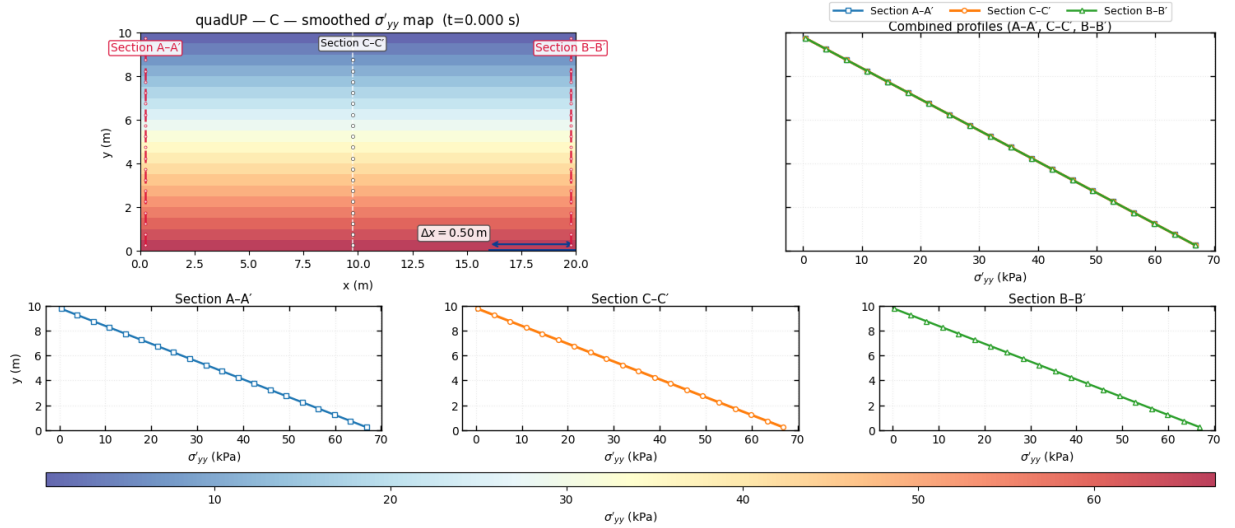
The corresponding seepage flux fields during Stage C are shown in Figures 5.51 and 5.52 for the New Element and QuadUP formulations, respectively. In both cases, the flux vectors are concentrated near the localized drain, defining a preferential flow path. The magnitude and spatial distribution of the flux obtained with the New Element closely match those of the reference QuadUP element.

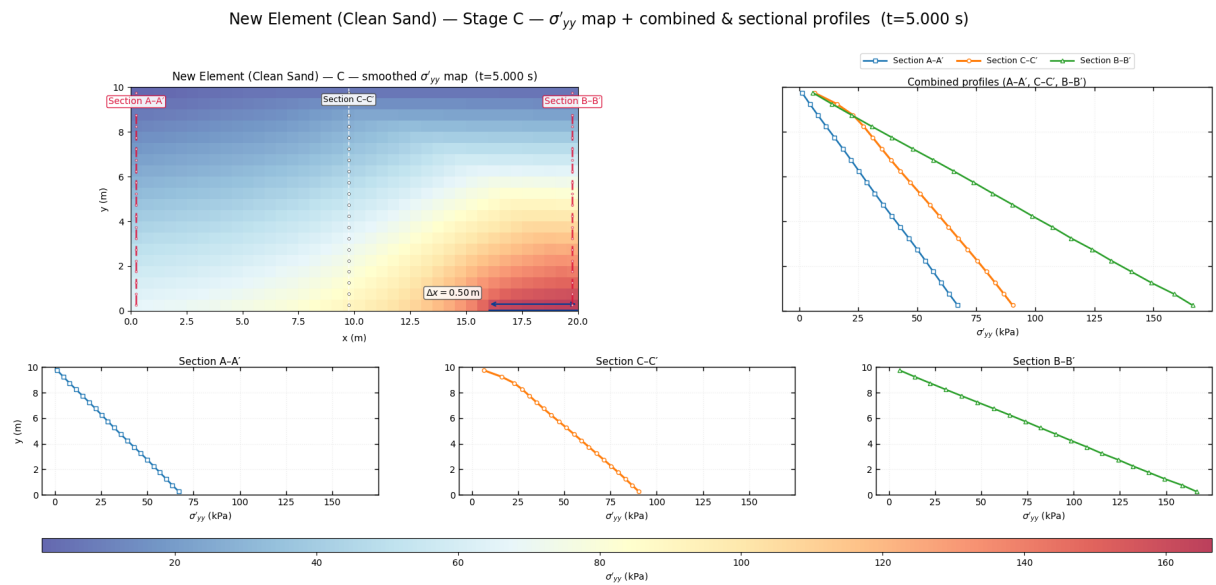
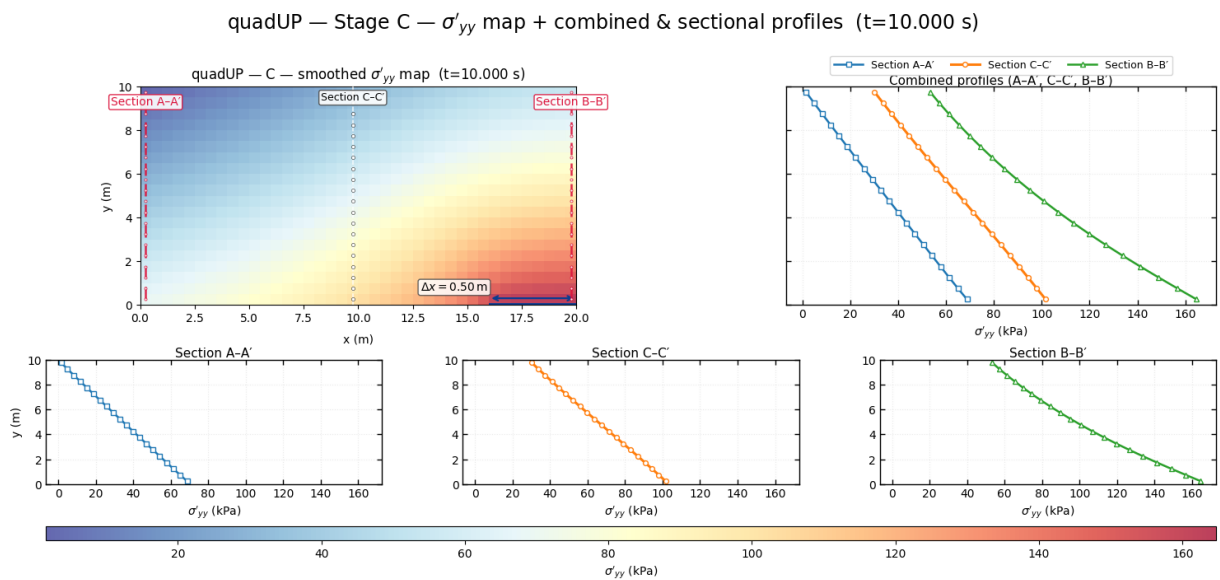
Figures 5.53 and 5.54 present the hydraulic gradient fields during Stage C for the New Element and QuadUP element, respectively. In both cases, high gradients develop near the drainage boundary and decrease smoothly away from the seepage front, with no numerical artifacts observed.

Finally, Figures 5.55 and 5.56 illustrate the effective horizontal and vertical permeability distributions computed using the New Element. The permeability remains close to its saturated value in fully saturated regions and decreases sharply within the partially saturated zone near the drain, coinciding with the desaturation front and confirming the correct implementation of the saturation-dependent permeability relationship.

Fig. 5.40 New Element (PW), $t = 0$ sFig. 5.41 QuadUP (PW), $t = 0$ s

Fig. 5.42 New Element (PW), $t = 5$ sFig. 5.43 QuadUP (PW), $t = 10$ s

New Element (Clean Sand) — Stage C — σ'_{yy} map + combined & sectional profiles ($t=0.000$ s)Fig. 5.44 New Element (Vertical Effective Stress), $t = 0$ squadUP — Stage C — σ'_{yy} map + combined & sectional profiles ($t=0.000$ s)Fig. 5.45 QuadUP (Vertical Effective Stress), $t = 0$ s

Fig. 5.46 New Element (Vertical Effective Stress), $t = 5$ sFig. 5.47 QuadUP (Vertical Effective Stress), $t = 10$ s

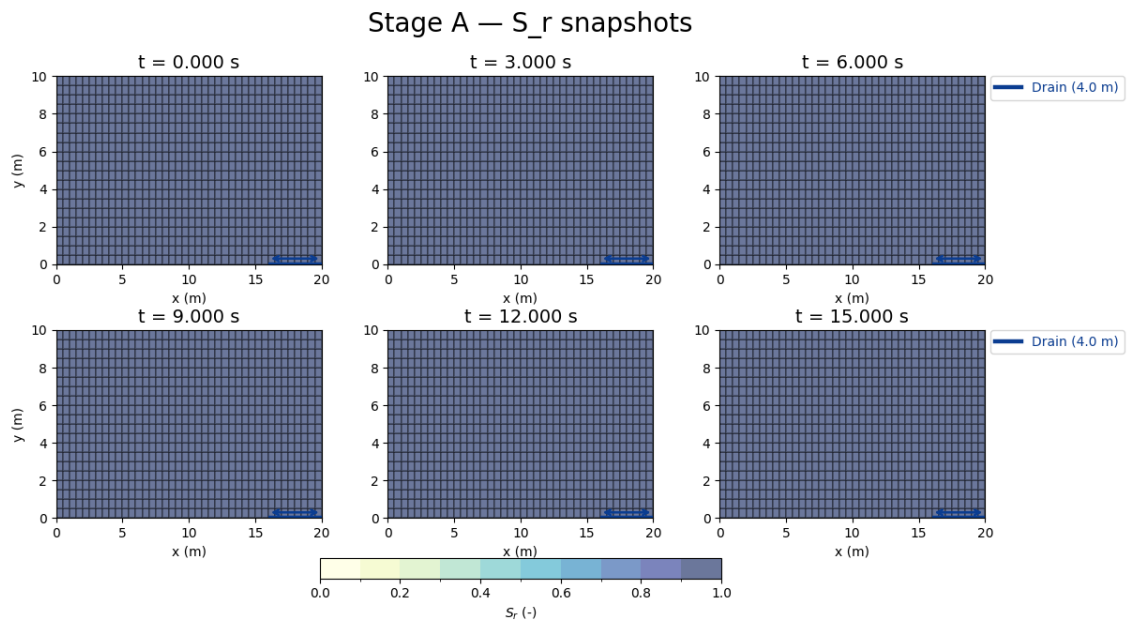


Fig. 5.48 New Element (Degree of Saturation), Stage A

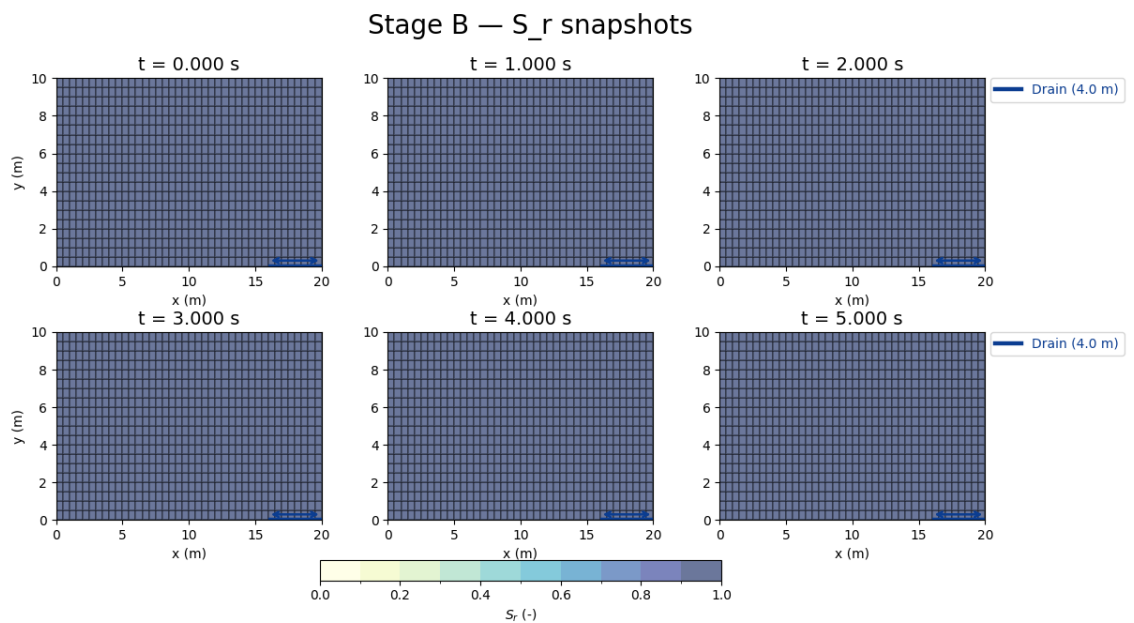


Fig. 5.49 New Element (Degree of Saturation), Stage B

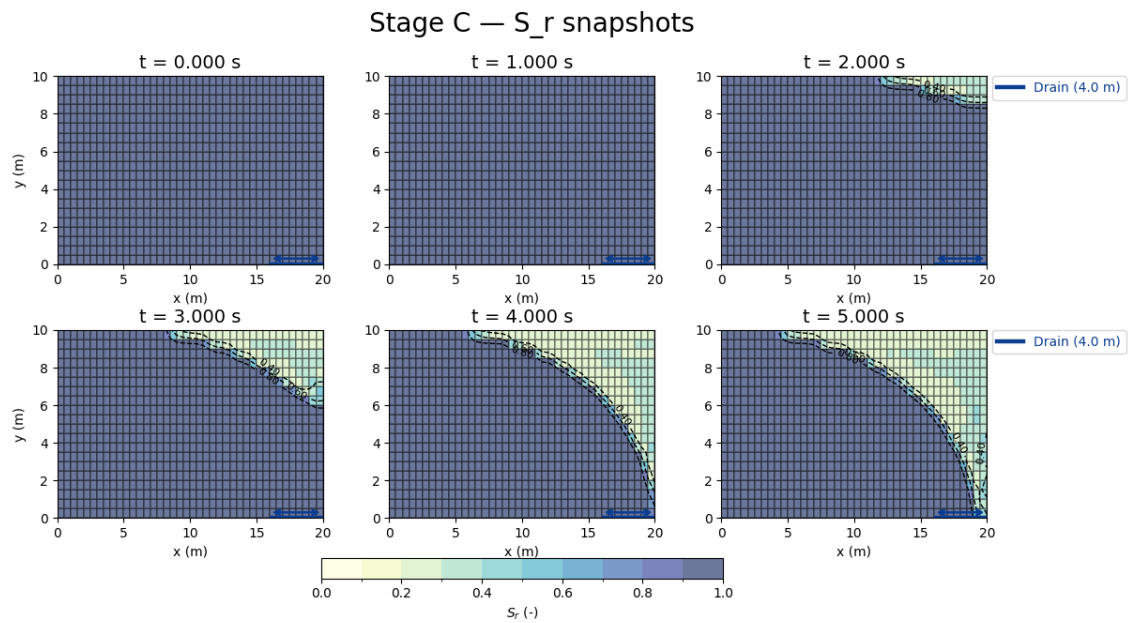


Fig. 5.50 New Element (Degree of Saturation), Stage C

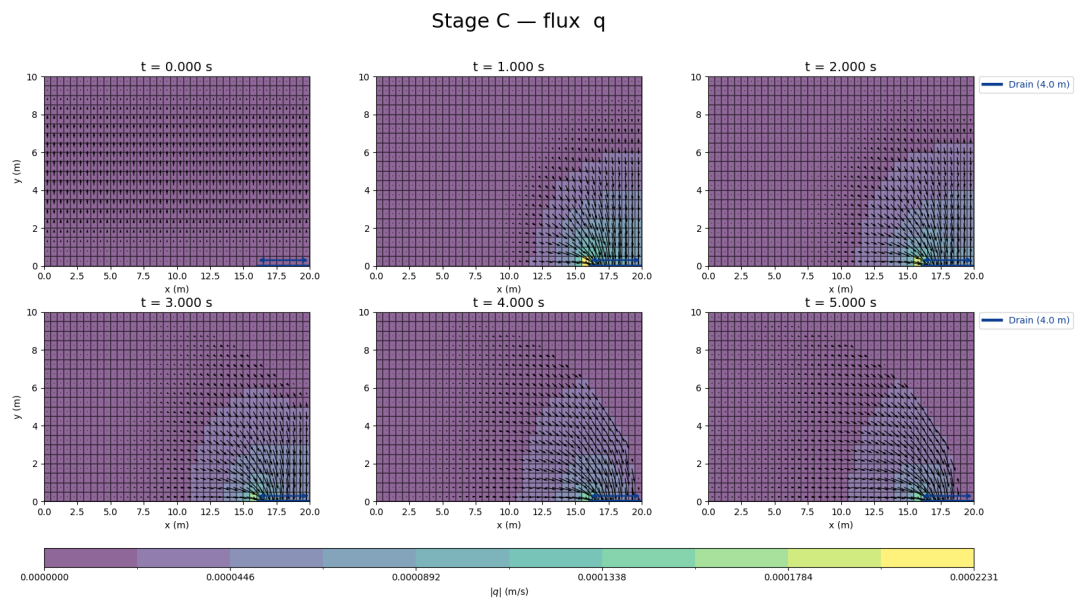


Fig. 5.51 New Element (Flux), Stage C

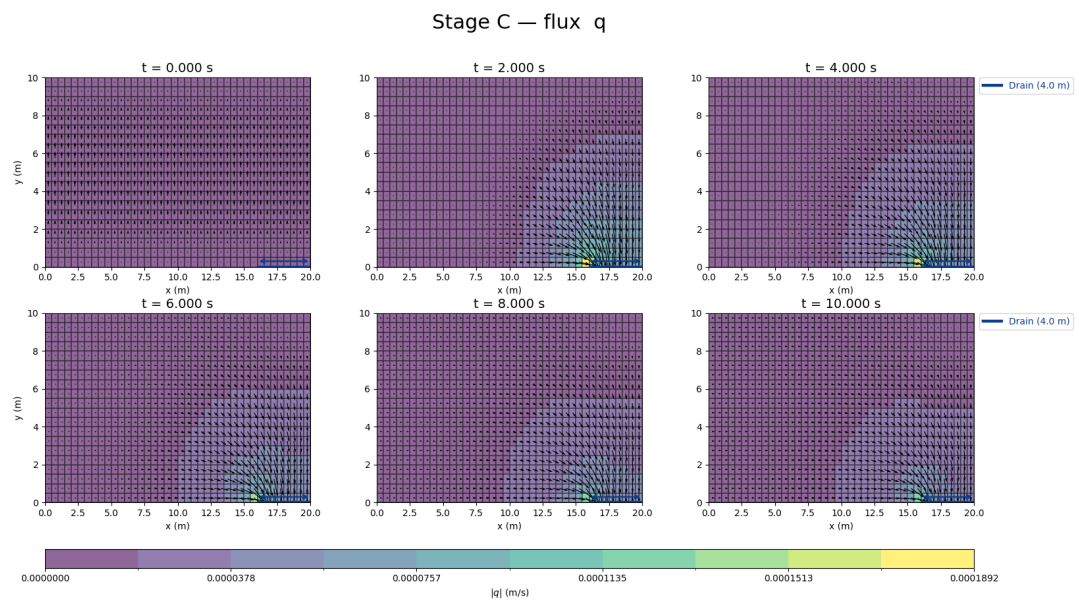


Fig. 5.52 QuadUP (Flux), Stage C

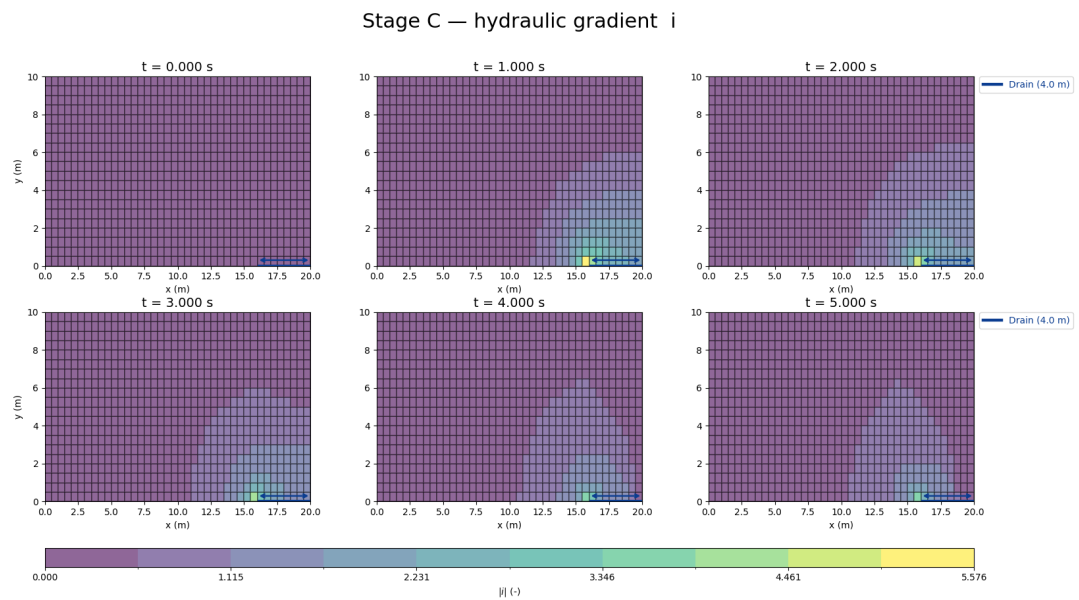


Fig. 5.53 New Element (Hydraulic Gradient), Stage C

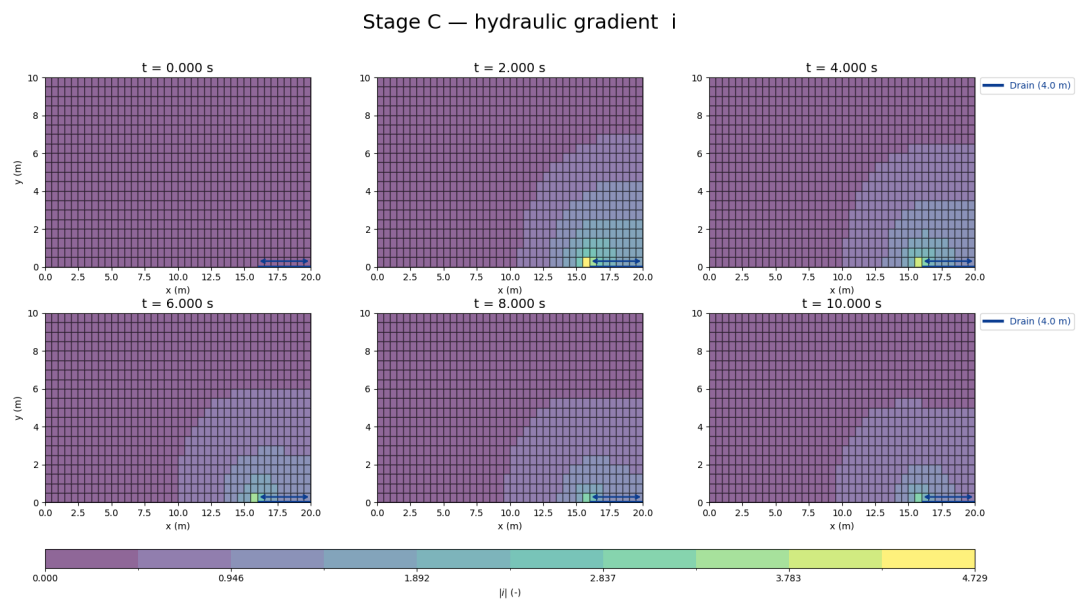


Fig. 5.54 QuadUP (Hydraulic Gradient), Stage C

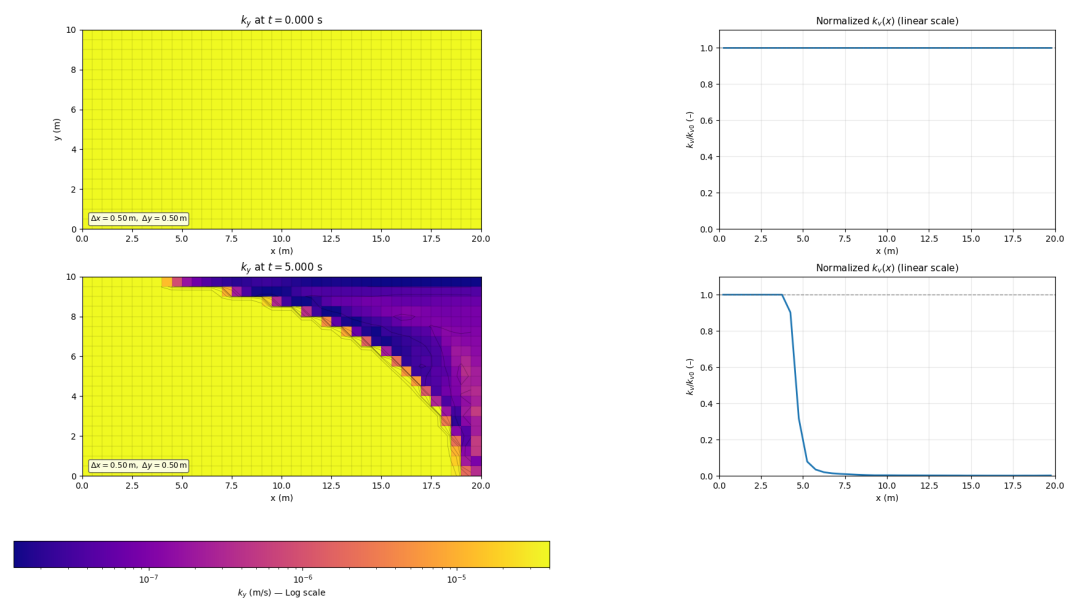


Fig. 5.55 New Element (Permeability vs. Effective Permeability, Horizontal)

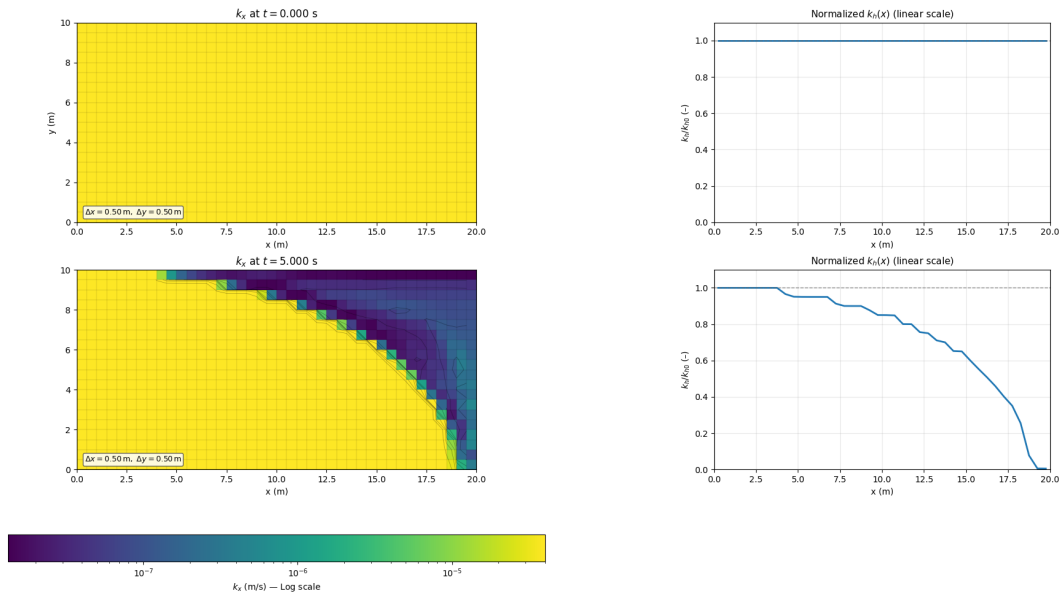


Fig. 5.56 New Element (Permeability vs. Effective Permeability, Vertical)

Finally, the effect of seepage and partial desaturation on the hydraulic response of the soil is examined through the pore–water pressure fields shown in Figure 5.57, comparing the results obtained with the New Element and the reference QuadUP formulation. As drainage is activated, pore–water pressure progressively decreases in the vicinity of the localized drain, defining the development of the seepage field during Stage C.

For the clean sand considered in this study, the New Element predicts only very low negative pore–water pressures near the drainage boundary. This response is physically consistent, as clean sands subjected to drainage are not expected to sustain significant suction; instead, the hydraulic response is primarily governed by partial desaturation rather than by the development of large negative pore–water pressures.

In contrast, the QuadUP formulation exhibits the development of pronounced negative pore–water pressures in the same region. Such values are not consistent with the expected hydraulic behavior of clean sands and reflect the inherent limitations of a saturated formulation when applied to seepage problems involving desaturation. The occurrence of unrealistically large negative pore–water pressures may lead to an overestimation of effective stresses and highlights the need for an unsaturated formulation capable of properly capturing phase changes during drainage.

Stage C — Pressure comparison: New Element vs quadUP (t0 & t_last)

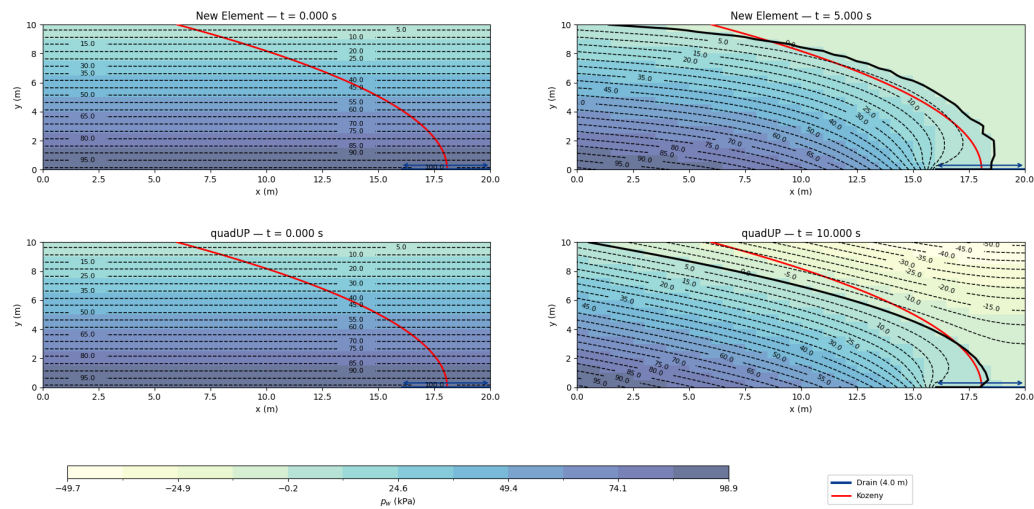
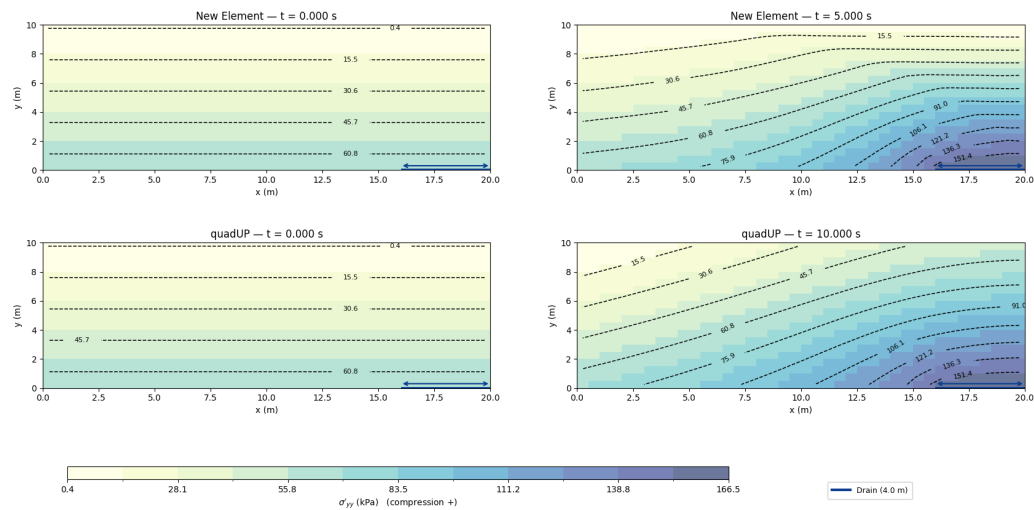
Fig. 5.57 New Element vs. QuadUP (p_v).Stage C — Effective stress σ'_{yy} : New Element vs quadUP (t0 & t_last)

Fig. 5.58 New Element vs. QuadUP (Vertical Effective Stress)

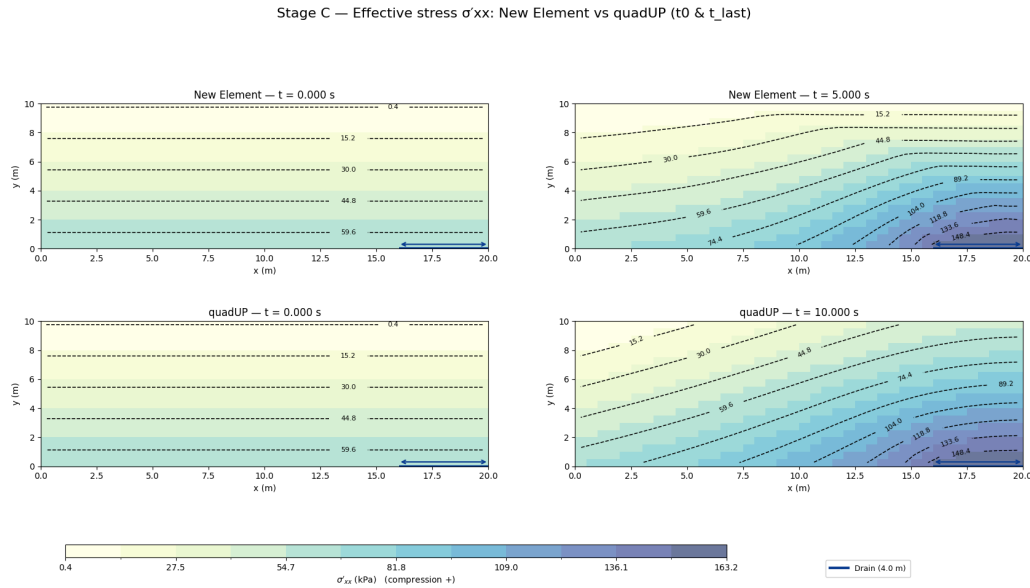


Fig. 5.59 New Element vs. QuadUP (Horizontal Effective Stress)

The corresponding mechanical response is illustrated through the vertical effective stress fields shown in Figure 5.58, which compares the results obtained with the New Element and the QuadUP formulation at the end of Stage C. As pore–water pressure dissipates near the drainage boundary, an increase in vertical effective stress is observed in the same region, reflecting the transfer of load from the fluid phase to the soil skeleton.

In addition, the distribution of the phreatic surface was verified analytically by applying the classical Kozeny method [78] for steady seepage. The corresponding analytical solution is superimposed in red in Figure 5.57, allowing a direct comparison with the numerical predictions. The close agreement confirms that both finite element formulations reproduce consistently the expected hydraulic head distribution and the resulting pore–pressure field.

The vertical stress contours obtained with both formulations exhibit similar spatial trends and comparable magnitudes, indicating that the proposed element accurately captures the stress redistribution induced by coupled seepage and drainage processes, despite the differences observed in the pore–pressure response.

The horizontal stress response under confined seepage conditions is shown in Figure 5.59, which presents the horizontal effective stress distributions computed using the New Element and the QuadUP formulation. At early stages, the horizontal stress field remains nearly uniform, reflecting the confined conditions and the absence of significant lateral deformation.

As seepage progresses, a gradual redistribution of horizontal stresses is observed near the drainage boundary, consistent with the local reduction in pore–water pressure and the associated increase in effective stress, while lateral equilibrium is preserved throughout the domain.

Taken together, the results presented in Figures 5.48 through 5.59 demonstrate that the New Element reliably reproduces the coupled evolution of pore–water pressure, degree of saturation, hydraulic flux, effective permeability, and stress during confined seepage with localized drainage. In particular, the absence of unrealistically large negative pore–water pressures in clean sand provides strong evidence of the physical consistency and robustness of the proposed finite element formulation under unsaturated flow conditions.

5.7 Seepage analysis through an earth dam with downstream toe drain

A two-dimensional seepage analysis was performed on an earth-dam-type embankment to verify the correct implementation of coupled hydro–mechanical boundary conditions and staged hydraulic loading in the QuadUP-based formulation implemented in OpenSees. The geometry represents a homogeneous embankment founded on an impervious base, with an upstream reservoir retaining water on the left side and a downstream toe drain located at the base of the right slope. The total length of the model is 240 m, and the dam height is approximately 20 m.

It is important to note that this example was not intended to reproduce the complete hydraulic regime of a real earth dam under operational reservoir conditions. Instead, it was designed as an idealized verification problem aimed at assessing the implementation of staged hydraulic boundary conditions and the coupled hydro–mechanical response of the proposed element. For this reason, the embankment was initially assumed to be fully saturated, and a hydrostatic pore–water pressure field was first established throughout the domain. The dam surfaces were then treated as no-flow boundaries, while drainage was activated only at the downstream toe. This simplified configuration provides a controlled hydraulic gradient and allows a direct comparison between the reference QuadUP formulation and the proposed New Element under identical boundary conditions.

The dam geometry, finite element discretization, and boundary conditions adopted during the staged seepage analysis are summarized in Figure 5.60. Two stages are distinguished: the initial gravity equilibrium stage, shown in Figure 5.60a, and the subsequent transient emptying stage, shown in Figure 5.60b.

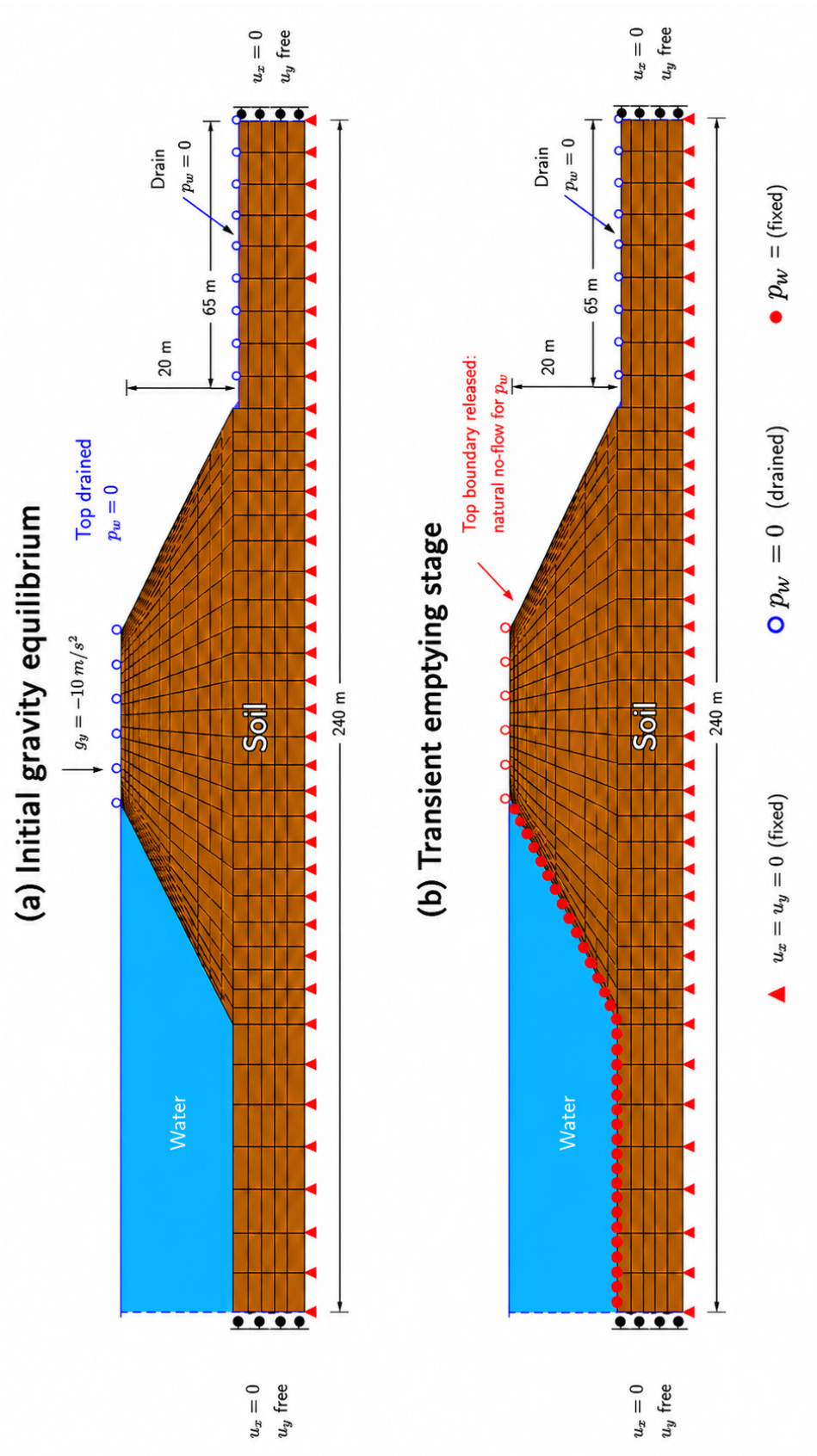


Fig. 5.60 Dam geometry, finite element mesh, and boundary conditions adopted in the staged seepage analysis: (a) initial gravity equilibrium stage and (b) transient emptying stage.

The finite element discretization consists of 1,217 nodes and 1,140 elements, providing a refined mesh in the dam body and near the drainage zone to accurately capture hydraulic gradients and pore-pressure variations. Pre-processing and post-processing were carried out using the finite element pre- and post-processing software GiD [26], together with the custom UW problem type developed by MacGann and Arduino [99], which automates the generation of OpenSees input files (Tcl scripts) and facilitates visualization of coupled hydraulic–mechanical responses.

Initial hydraulic condition. The analysis was initialized under fully saturated conditions throughout the entire domain. This assumption was adopted for verification purposes and should not be interpreted as a realistic representation of the hydraulic regime of an operating earth dam.

Rather, it provides a controlled initial condition from which the effect of the downstream toe drain can be isolated. A hydrostatic pore-water pressure distribution was first established prior to the seepage stage, ensuring that the initial effective stress state corresponds to a saturated embankment subjected to gravity loading only.

Constitutive model and fluid phase. The soil skeleton was modeled as linear elastic during the gravity initialization phase. The elastic, hydraulic, and soil–water retention properties adopted in the analysis correspond to those summarized in Tables 5.1 and 5.3. These include the intrinsic saturated permeability and the van Genuchten parameters governing the suction–saturation relationship. The analysis was performed using both the reference QuadUP element and the proposed quadUPSVG element, allowing a direct comparison of their responses under identical hydraulic boundary conditions.

To enhance numerical stability during the gravity initialization stage, the intrinsic permeabilities were temporarily set equal to unity ($k_x = k_y = 1.0$ m/s), thereby promoting rapid dissipation of excess pore pressures. Prior to initiating the seepage analysis, the permeabilities were restored to the isotropic values reported in Table 5.1, ensuring a physically consistent hydraulic response during the subsequent drainage stages.

Mechanical boundary conditions. The mechanical boundary conditions adopted in the two stages are indicated in Figure 5.60. During the initial gravity equilibrium stage (Figure 5.60a), the base of the model was fully fixed by imposing $u_x = u_y = 0$, while the lateral boundaries were constrained in the horizontal direction, $u_x = 0$, and left free in the vertical direction. These lateral constraints were applied not only to the vertical truncation boundaries

of the foundation soil, but also to the lateral faces of the embankment during the gravity initialization stage.

After the initial stress state was established, the reaction forces at the constrained lateral nodes were recorded and re-applied as equivalent nodal forces before releasing the corresponding horizontal restraints. This procedure was adopted to preserve static equilibrium and avoid spurious stress redistribution associated with the sudden removal of the artificial lateral constraints. In the subsequent transient emptying stage (Figure 5.60b), the base fixity was maintained, while the lateral horizontal restraints were released after applying the equivalent nodal reaction forces required to preserve the stress state generated during initialization.

In this way, the release procedure was not restricted to the foundation boundaries alone, but was also applied to the lateral faces of the dam body where the corresponding temporary horizontal constraints had been introduced during initialization. In the seepage stage, the downstream dam-face load pattern was removed, so that the downstream face could respond consistently with the activated toe-drain boundary condition, whereas the remaining equivalent boundary loads were retained to preserve the initial stress state.

Hydraulic boundary conditions. The hydraulic boundary conditions were imposed through the pore-water pressure degree of freedom of the coupled u - p_w formulation. In the initial gravity equilibrium stage (Figure 5.60a), selected crest nodes were temporarily assigned drained conditions, $p_w = 0$, in order to define the reference hydrostatic pore-water pressure field. These constraints were subsequently removed before the seepage stage.

During the transient emptying stage (Figure 5.60b), the hydraulic boundary conditions were modified. The upstream water-contact boundary was prescribed by maintaining the pore-water pressure values obtained at the end of the initialization stage; this condition is represented by the red filled markers in Figure 5.60b. The downstream toe drain was prescribed as $p_w = 0$, represented by the blue open circles. All remaining external soil boundaries, including the exposed dam surfaces and the foundation base, were treated as natural no-flow boundaries, since no pore-water pressure condition was imposed there.

Therefore, the hydraulic boundary conditions applied to the entire domain can be summarized as follows: the upstream water-contact boundary was kept at the initial pore-water pressure values, the downstream toe drain was assigned $p_w = 0$, and all other external soil boundaries were left as natural no-flow boundaries for the pore-water pressure field.

Staged procedure.

1. **Gravity initialization (elastic):** transient integration was carried out to establish the self-weight stress state with elastic soil response. During this stage, the base was

fixed, the lateral boundaries were temporarily constrained in the horizontal direction, and selected crest nodes were drained to define the reference hydrostatic pore-water pressure field.

2. **Permeability update:** after gravity equilibrium was reached, the element permeabilities were updated from the temporary high values used for numerical stabilization to the target values adopted for the seepage analysis.
3. **Reaction-equivalent loading and release:** with the lateral u_x constraints still active, the boundary reactions were recorded and re-applied as equivalent nodal loads on the corresponding dam faces and lateral boundary columns. The horizontal restraints were then released to obtain a mechanically consistent condition without introducing spurious stress transients.
4. **Seepage stage with toe drain:** the upstream water-contact boundary was kept at the pore-water pressure values obtained at the end of the initialization stage, while the downstream toe drain was assigned $p_w = 0$ through `MultipleSupport` in $DOF = 3$. The remaining external boundaries were left as natural no-flow boundaries, and the system was then advanced in time using the implicit `BackwardEuler` scheme to observe pore-pressure redistribution and drainage-induced seepage.

Figure 5.61(a) presents the pore-water pressure distribution obtained under hydrostatic conditions after the gravity initialization stage using the proposed New Element. The pressure field exhibits the expected linear increase with depth, with equipotential lines approximately parallel to the upstream and downstream boundaries. This result confirms that the initial hydraulic state is correctly established and that the gravity loading and hydraulic boundary conditions are consistently implemented prior to the seepage stage.

The pore-water pressure fields obtained at the final step of the seepage analysis is shown in Figure 5.61 (b), respectively. In both cases, the presence of the downstream toe drain induces a clear hydraulic gradient directed toward the drainage boundary, resulting in a progressive reduction of pore-water pressure within the dam body. The pressure contours exhibit a smooth spatial variation, with lower pressures concentrated near the downstream toe and higher pressures maintained in the upstream portion of the embankment.

The horizontal and vertical permeability distributions obtained from the analysis are illustrated in Figures 5.62 (a) and (b). As expected, the permeability field remains spatially uniform within the dam body, confirming that the observed variations in pore-water pressure and flow patterns are driven by the imposed hydraulic boundary conditions rather than by artificial heterogeneity in material properties.

Figure 5.63 presents the degree of saturation obtained with the New Element at the final stage of the seepage analysis. A partially saturated zone develops in the vicinity of the downstream toe drain, where pore-water pressure is reduced to atmospheric conditions. The desaturation front propagates locally into the dam body, while the majority of the embankment remains close to full saturation. This behavior is consistent with the localized nature of the drainage boundary and confirms the correct coupling between pore-water pressure reduction and saturation evolution.

The seepage flux field computed with the New Element is shown in Figure 5.64. The flux vectors are oriented toward the downstream toe drain, defining a preferential flow path within the dam. Higher flux magnitudes are observed near the drainage boundary, while flux intensity decreases with distance from the drain, reflecting the spatial distribution of the hydraulic gradient. This flux pattern is fully consistent with the pore-water pressure and saturation fields presented previously.

Figures 5.65 and 5.69 show the horizontal and vertical stress distributions, respectively, at selected stages of the analysis. The stress fields evolve smoothly during the seepage process, with stress redistribution occurring in response to changes in pore-water pressure. The absence of spurious stress concentrations and the continuity of the stress contours across the dam body indicate a stable and mechanically consistent response.

The corresponding shear stress distribution is illustrated in Figure 5.70. Shear stresses are primarily concentrated in the sloping regions of the embankment, where geometric effects dominate the stress state. The overall distribution is physically reasonable and remains consistent throughout the analysis.

Figures 5.71 and 5.72 present the horizontal and vertical strain distributions, respectively. Strains remain relatively small and localized, with the largest values occurring in regions influenced by seepage-induced stress redistribution. The strain patterns evolve smoothly in time and remain compatible with the corresponding stress fields, indicating a coherent mechanical response.

An analytical evaluation of the degree of saturation and hydraulic conductivity was performed based on direct calculations using the formulations introduced earlier in this document. Figure 5.66 presents the degree of saturation, S_r , while Figures 5.67 and 5.68 show the corresponding horizontal and vertical permeability distributions, respectively. The comparison is carried out along a vertical section located near the crest. In all cases, the numerical results exhibit excellent agreement with the analytically derived profiles, demonstrating that the proposed element accurately captures both the van Genuchten soil–water retention relationship and the associated unsaturated permeability functions. The reference QuadUP formulation is not considered in this analysis.

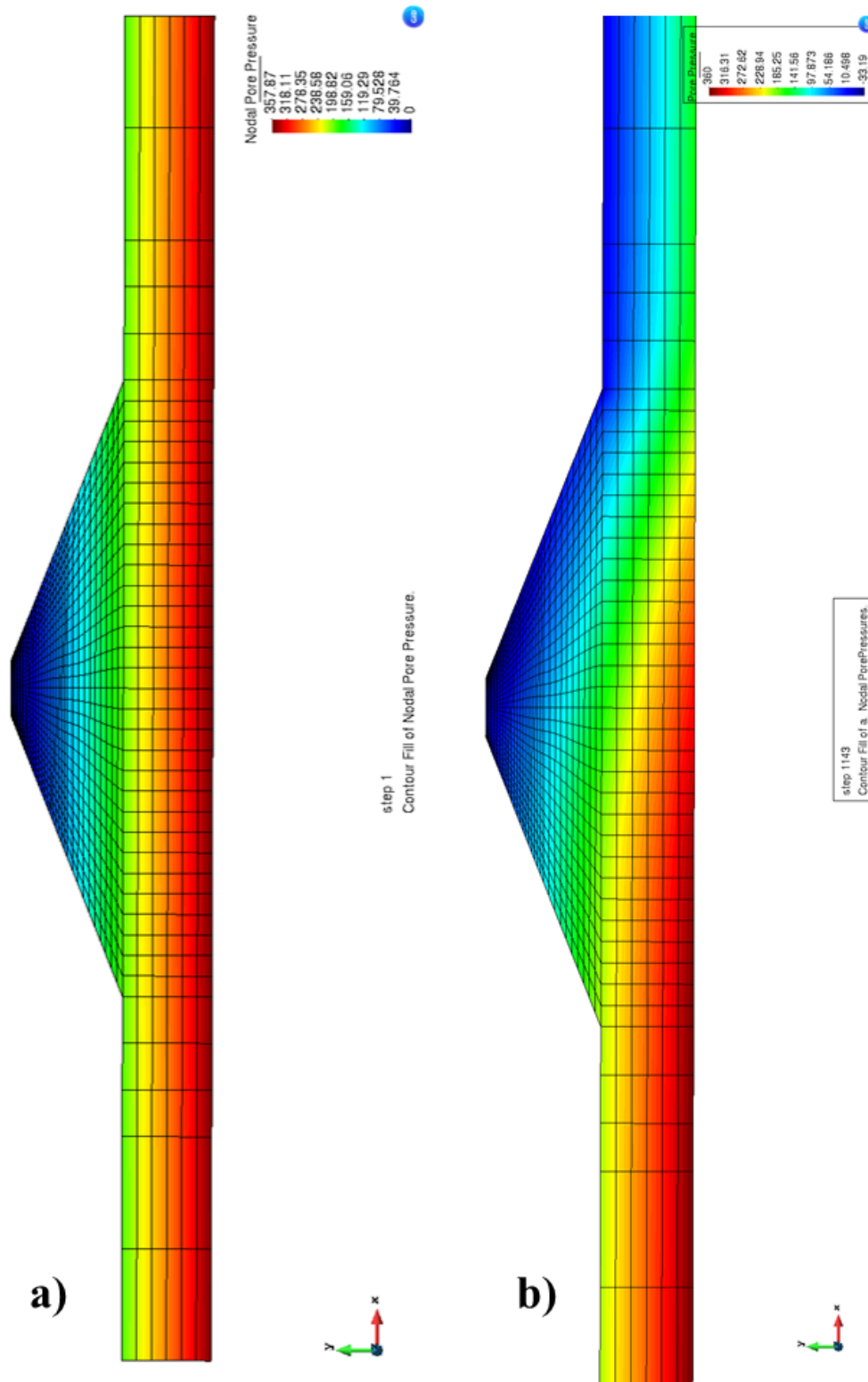


Fig. 5.61 Dam pore-water pressure – New Element ((a) Initial vs. (b) Last Step)

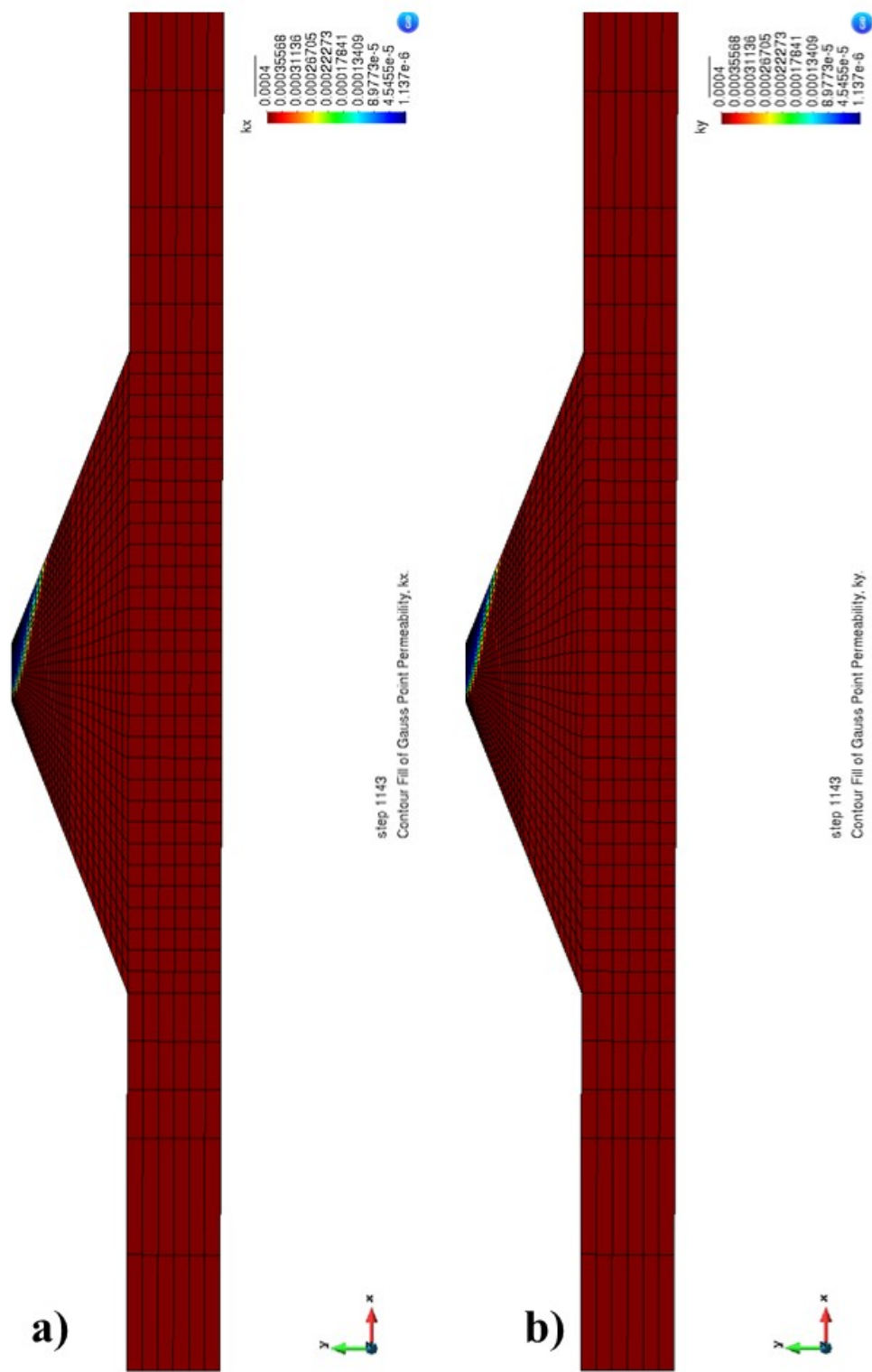


Fig. 5.62 (a) Horizontal and (b) vertical permeability distribution - New Element

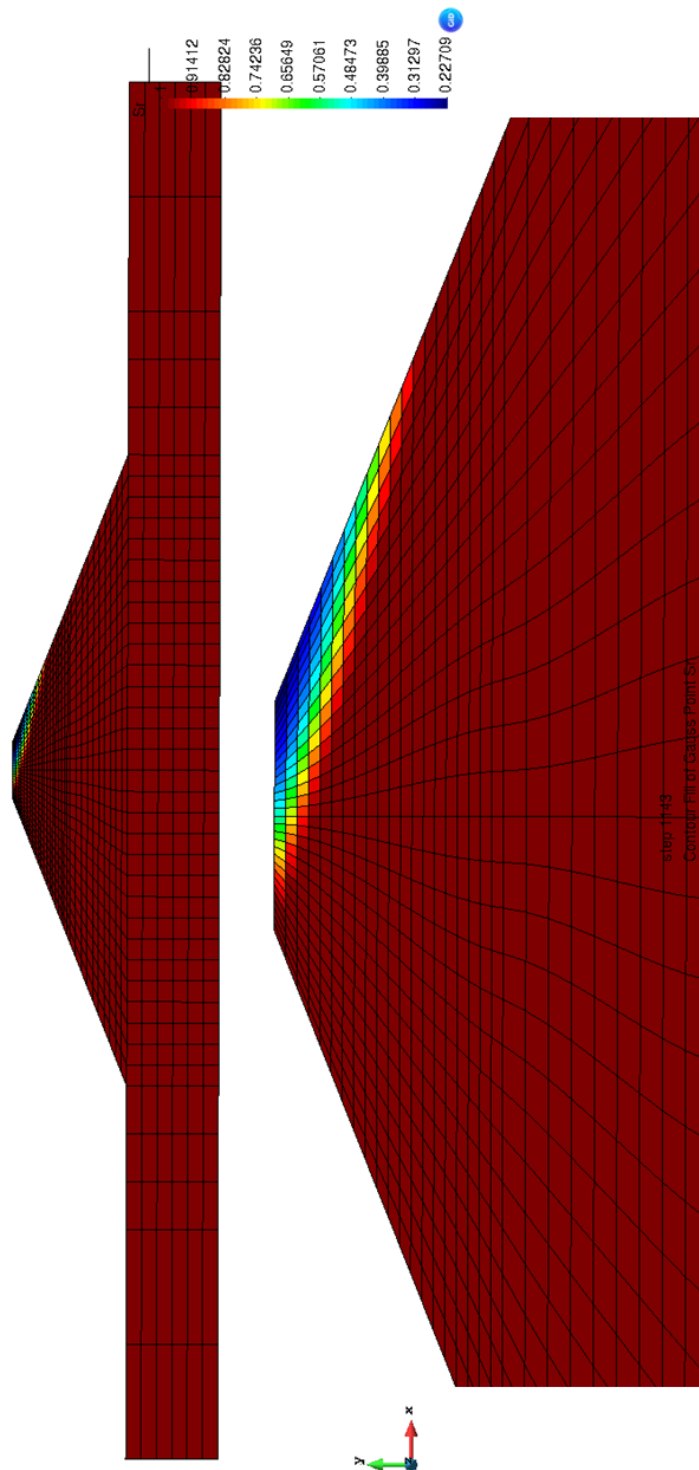


Fig. 5.63 Dam pore degree of saturation – New Element

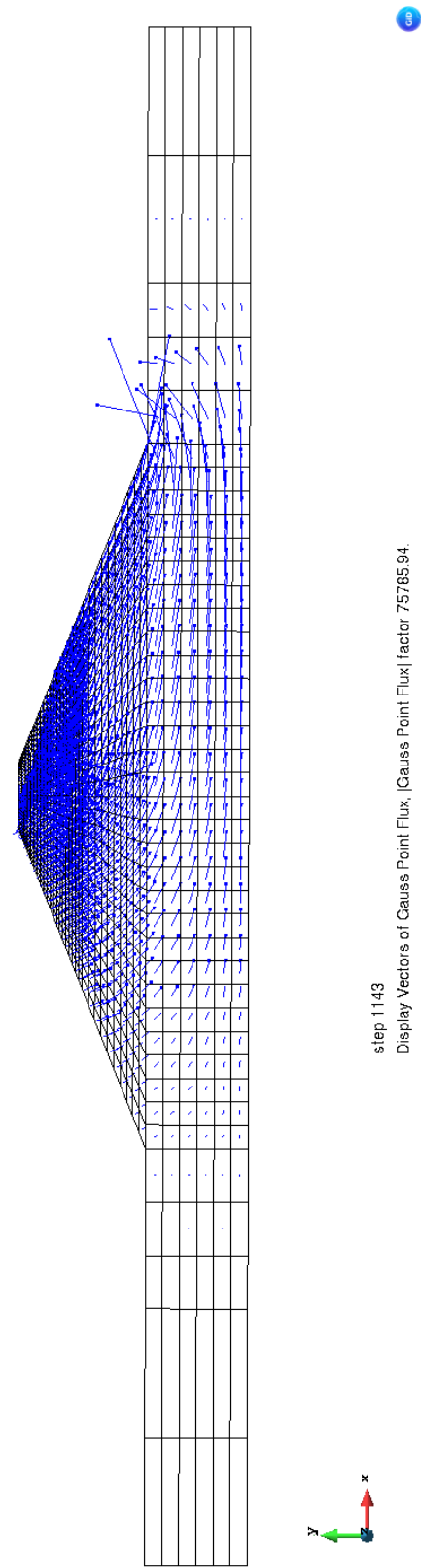


Fig. 5.64 Dam seepage -New Element

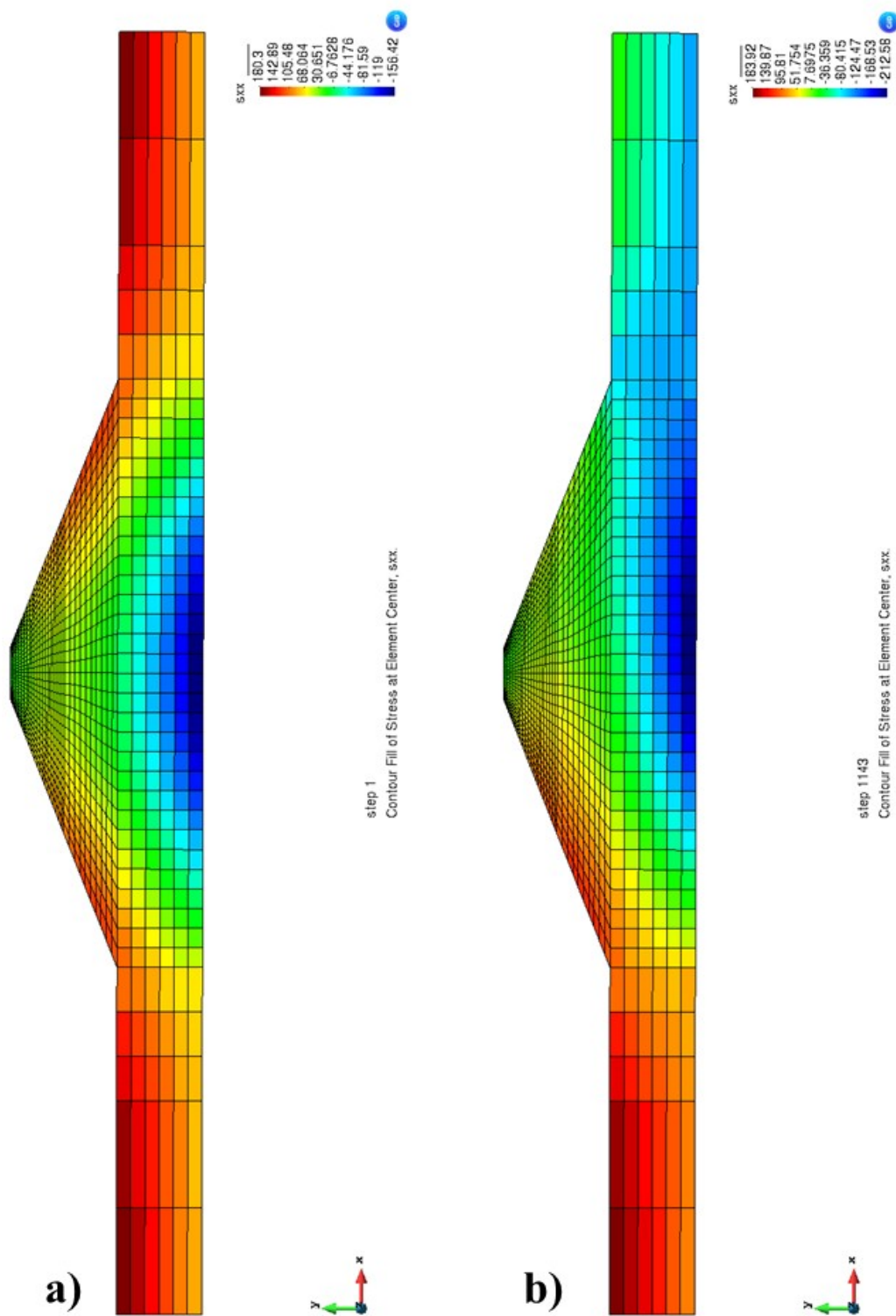


Fig. 5.65 Horizontal Effective Stresses at (a) first and (b) last step

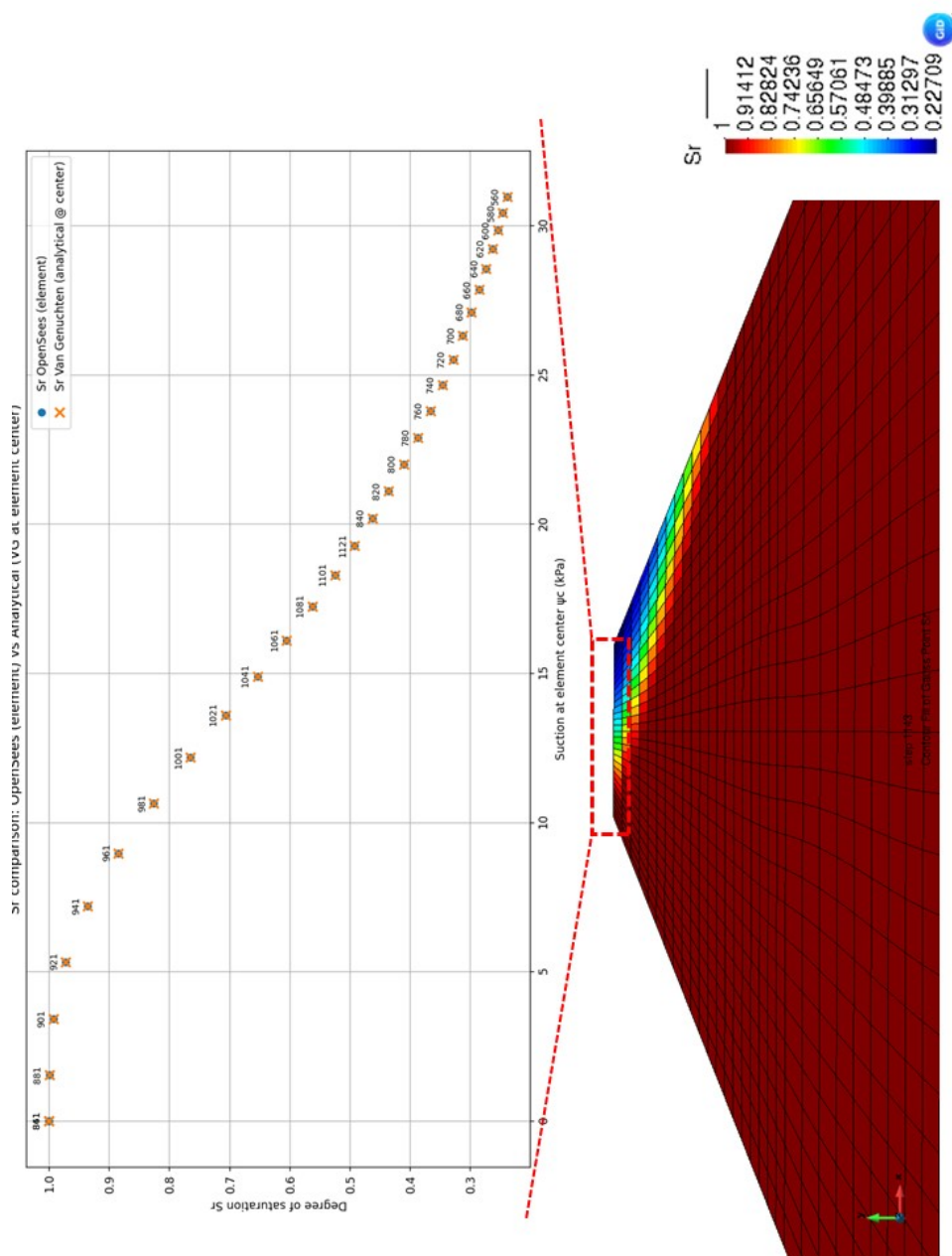


Fig. 5.66 Degree of Saturation OpenSees vs Analytical Solution

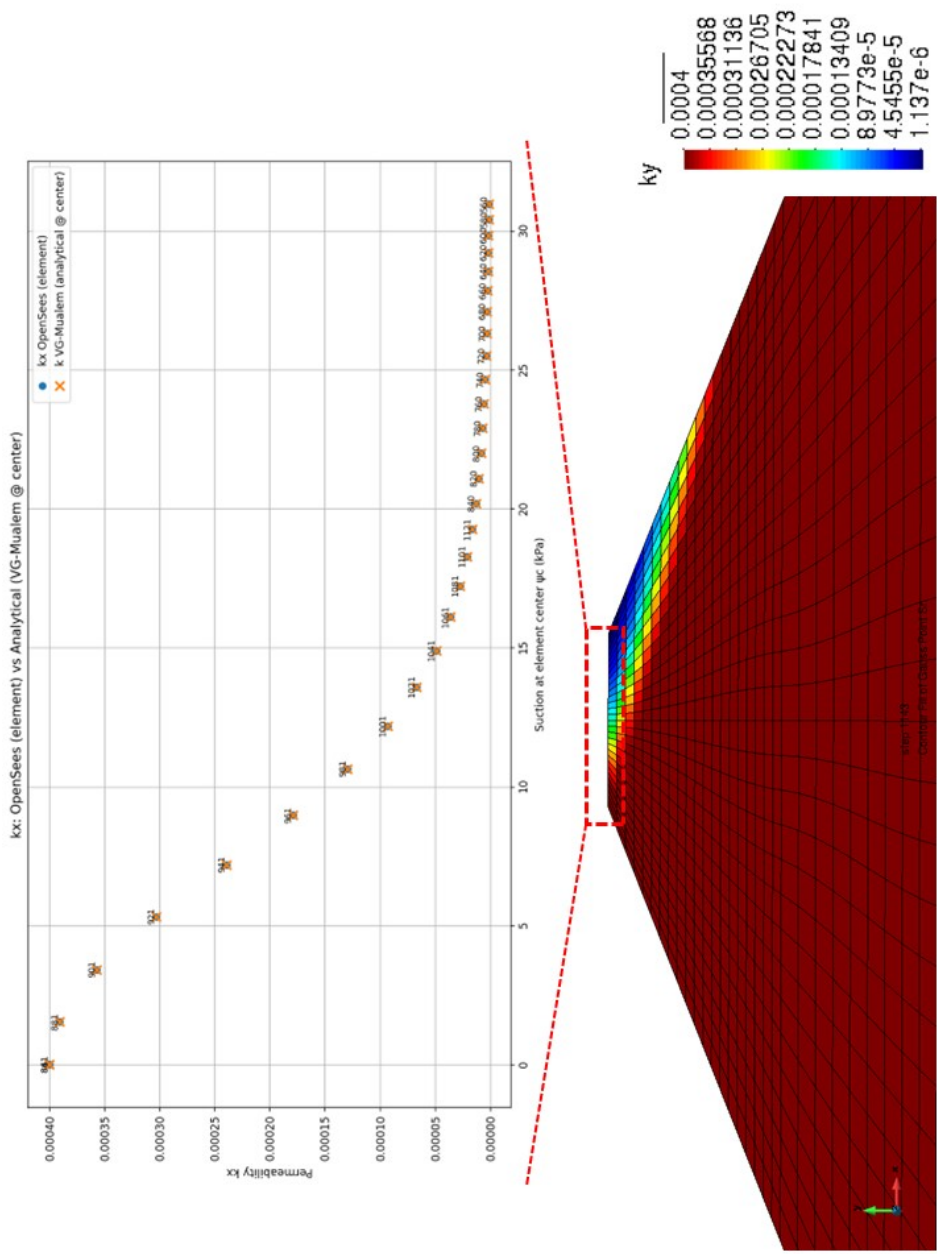


Fig. 5.67 Horizontal permeability OpenSees vs Analytical Solution

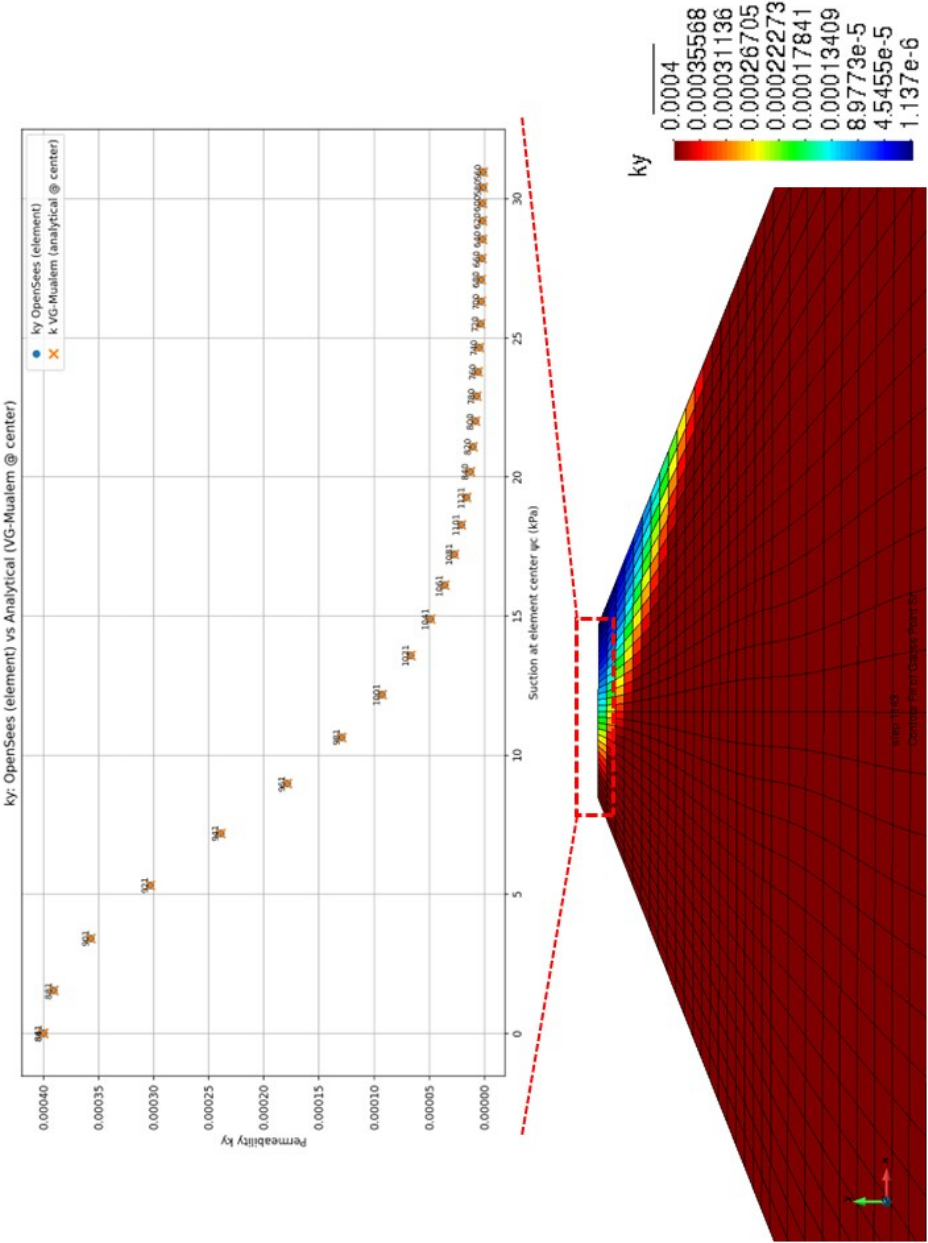


Fig. 5.68 Vertical permeability OpenSees vs Analytical Solution

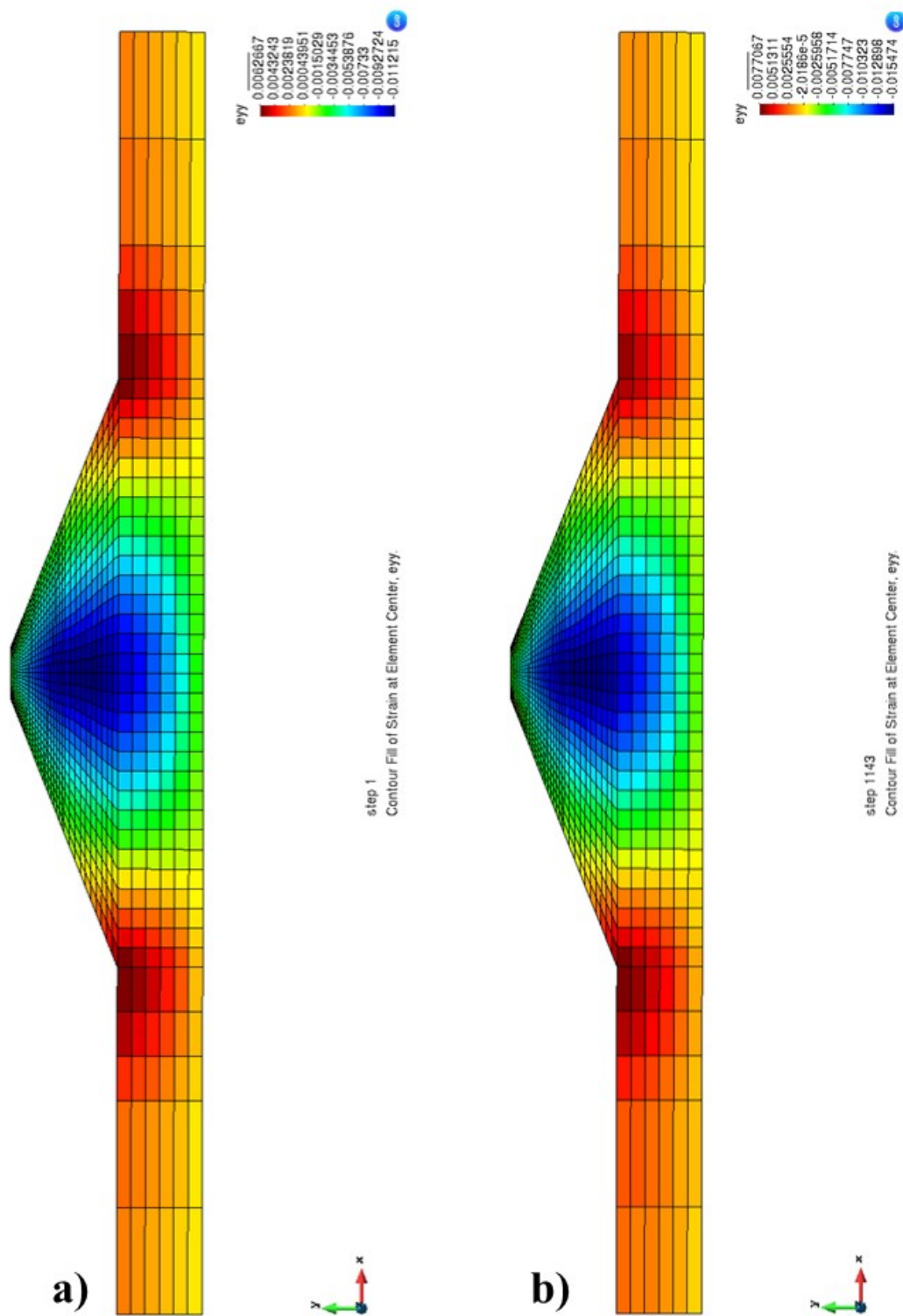


Fig. 5.69 Vertical Effective Stresses at (a) first and (b) last step

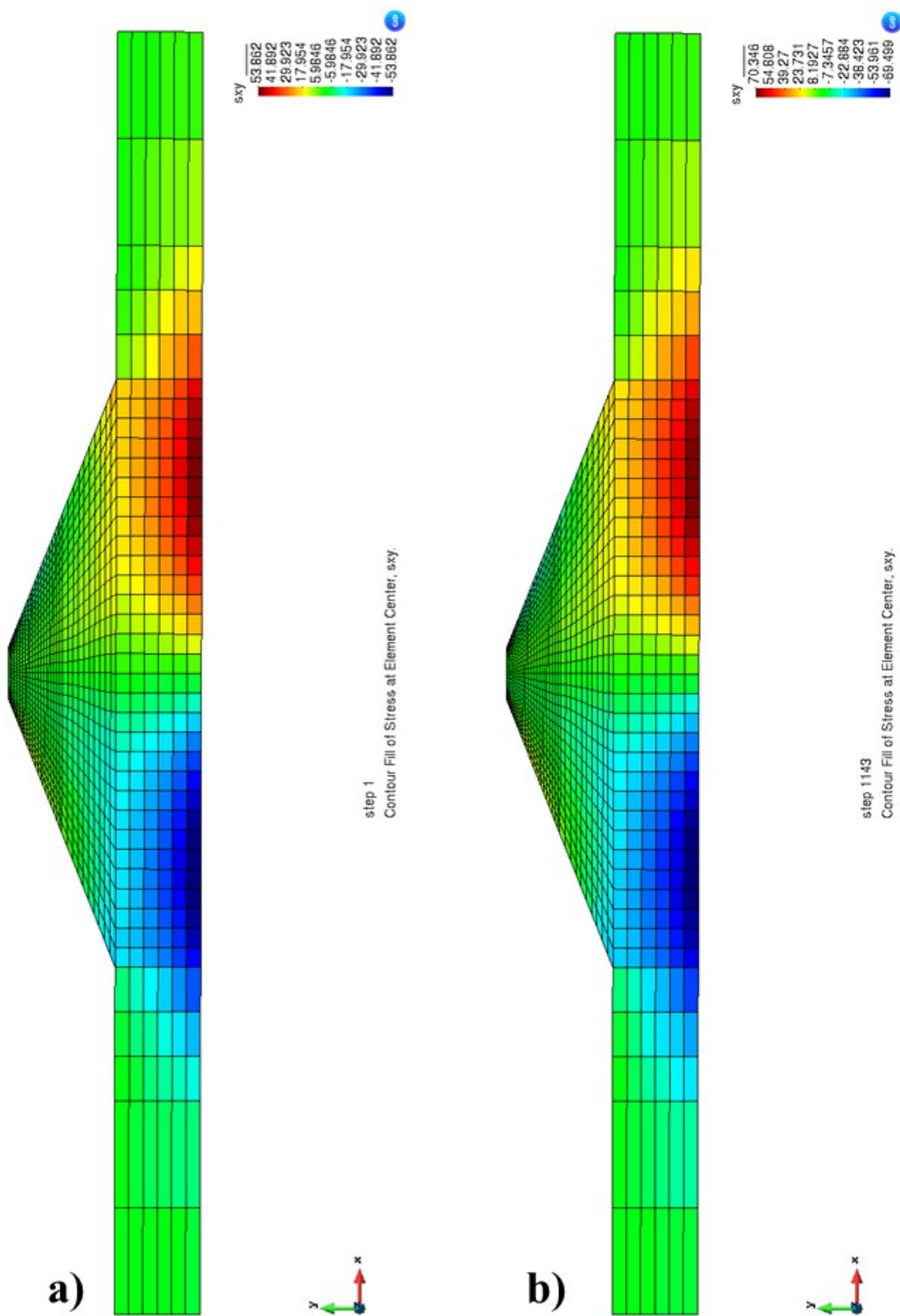


Fig. 5.70 Shear stresses at (a) first and (b) last step

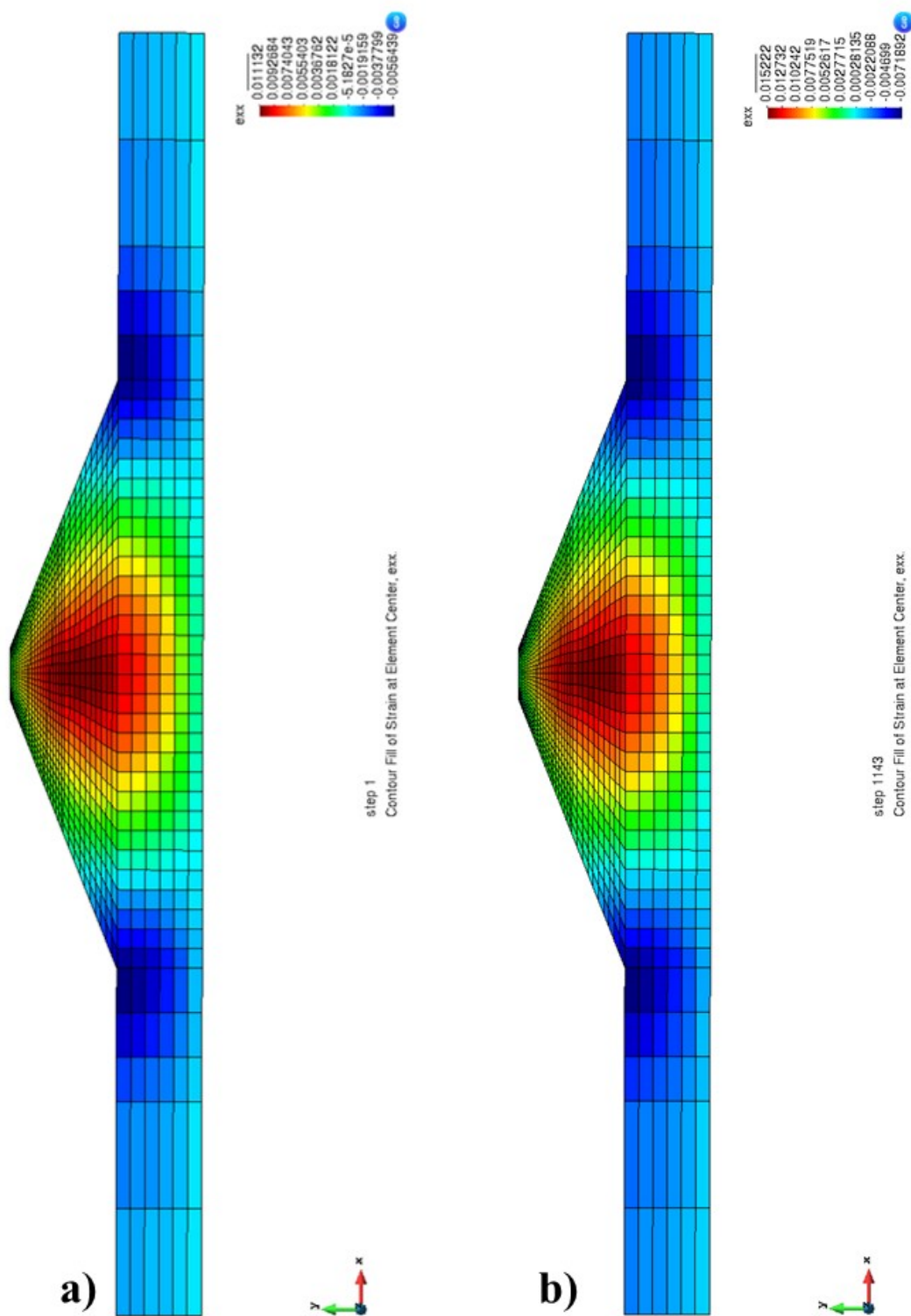


Fig. 5.71 Horizontal strains at (a) first and (b) last step

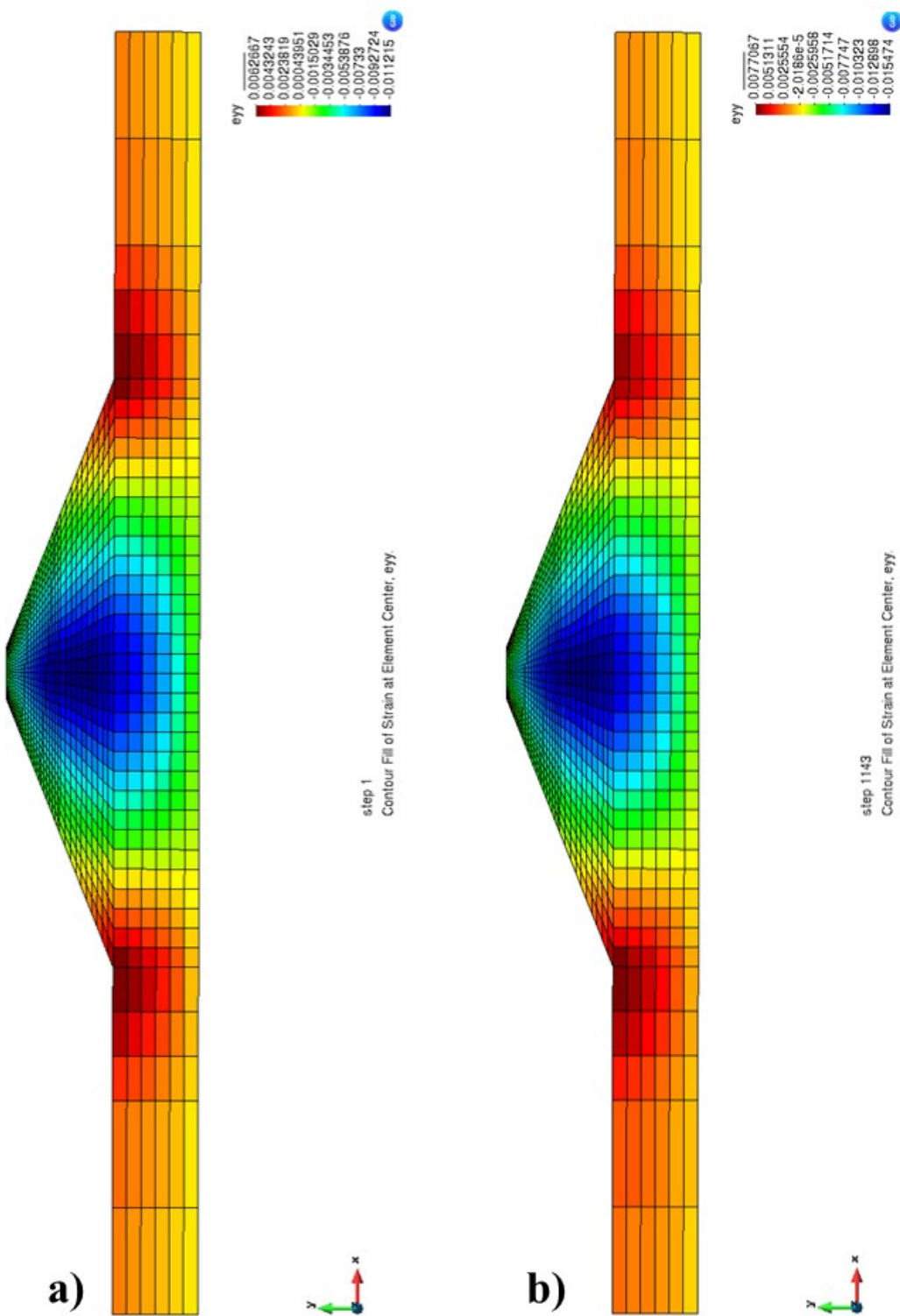


Fig. 5.72 Vertical strains at (a) first and (b) last step

Minor deviations near the boundaries are attributed to discretization effects. Overall, the agreement confirms the correct implementation of the SWRC and permeability formulation in the unsaturated zone of the embankment.

Overall, the results presented in Figures 5.61 through 5.72 demonstrate that the proposed New Element is able to consistently reproduce the coupled hydro-mechanical response of an earth dam subjected to seepage with a downstream toe drain. The close agreement with the reference QuadUP formulation in terms of pore-water pressure distributions, together with the physically consistent prediction of saturation evolution, seepage fluxes, stresses, and strains, provides strong verification of the correctness and robustness of the implemented formulation. This example confirms the suitability of the proposed element for practical seepage analyses in geotechnical engineering.

5.8 Seepage through an earth dam with downstream base drain

This verification example follows the same numerical framework presented in Section 5.2.4, including the constitutive assumptions, staged solution strategy, and implementation details adopted for the seepage analysis of an earth dam with a downstream toe drain. The same New Element implemented in OpenSees, finite element discretization, gravity initialization procedure, permeability updating strategy, and treatment of mechanical boundary conditions are employed herein and are therefore not repeated for brevity.

In contrast to the previous case, the embankment is assumed to be fully saturated throughout the analysis and does not retain an upstream reservoir. The key difference lies in the hydraulic boundary conditions applied during the seepage stage. Instead of a localized downstream toe drain, drainage is imposed along the entire base of the domain by prescribing a zero pore-water pressure condition, $p_w = 0$, at all basal nodes. This boundary condition is enforced using a MultipleSupport pattern applied to the pore-water pressure degree of freedom, $DOF = 3$, while all other external boundaries are treated as natural no-flow boundaries.

The dam geometry, finite element mesh, and boundary conditions adopted in this analysis are summarized in Figure 5.73. In the initial gravity equilibrium stage, Figure 5.73(a), the base is mechanically fixed by imposing $u_x = u_y = 0$, while the lateral boundaries are constrained in the horizontal direction and left free vertically. A drained condition, $p_w = 0$, is imposed at the crest to establish the initial hydrostatic pore-water pressure field.

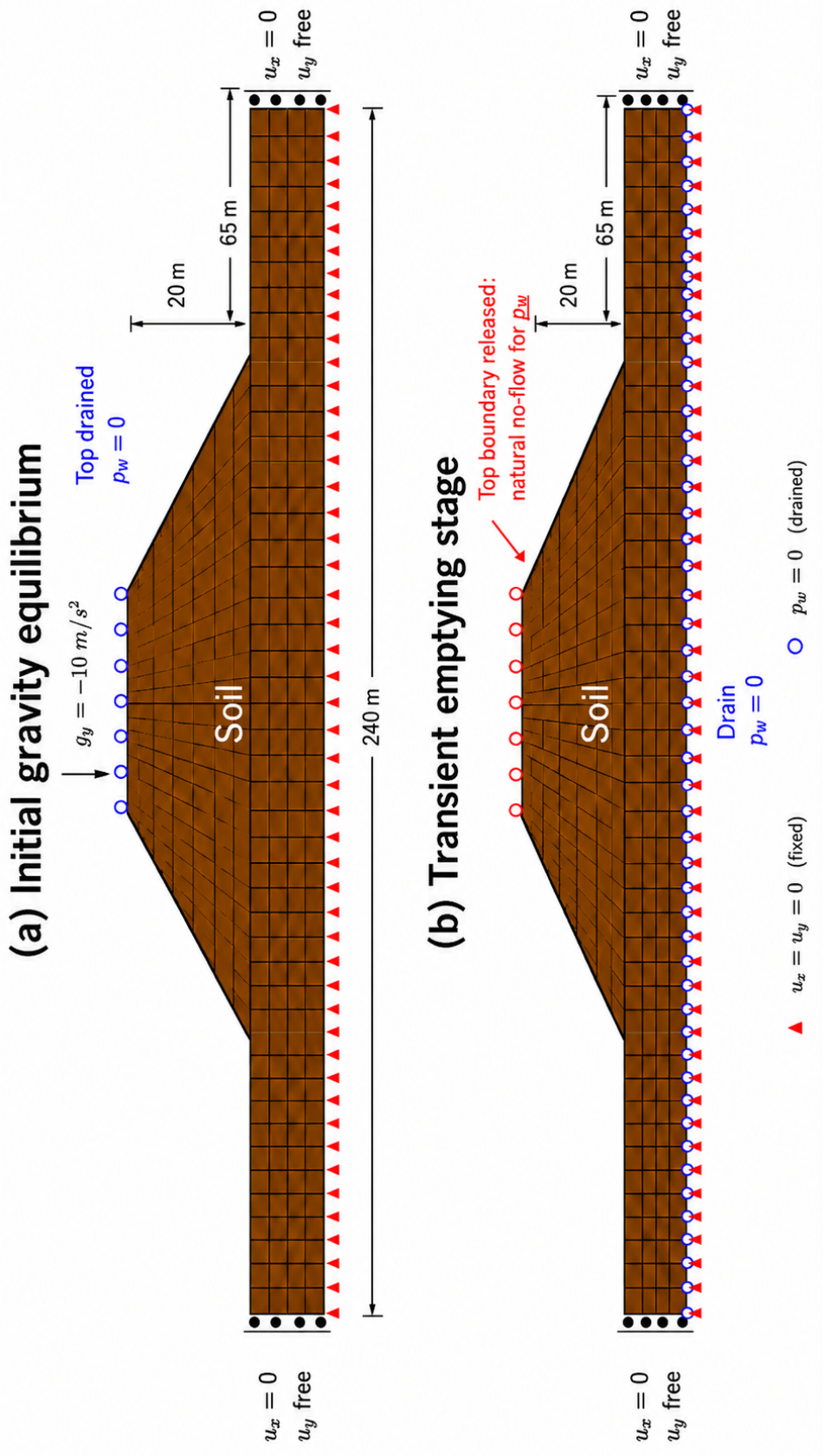


Fig. 5.73 Dam geometry, finite element mesh, and boundary conditions adopted in the staged seepage analysis with full-base drainage: (a) initial gravity equilibrium stage and (b) transient emptying stage.

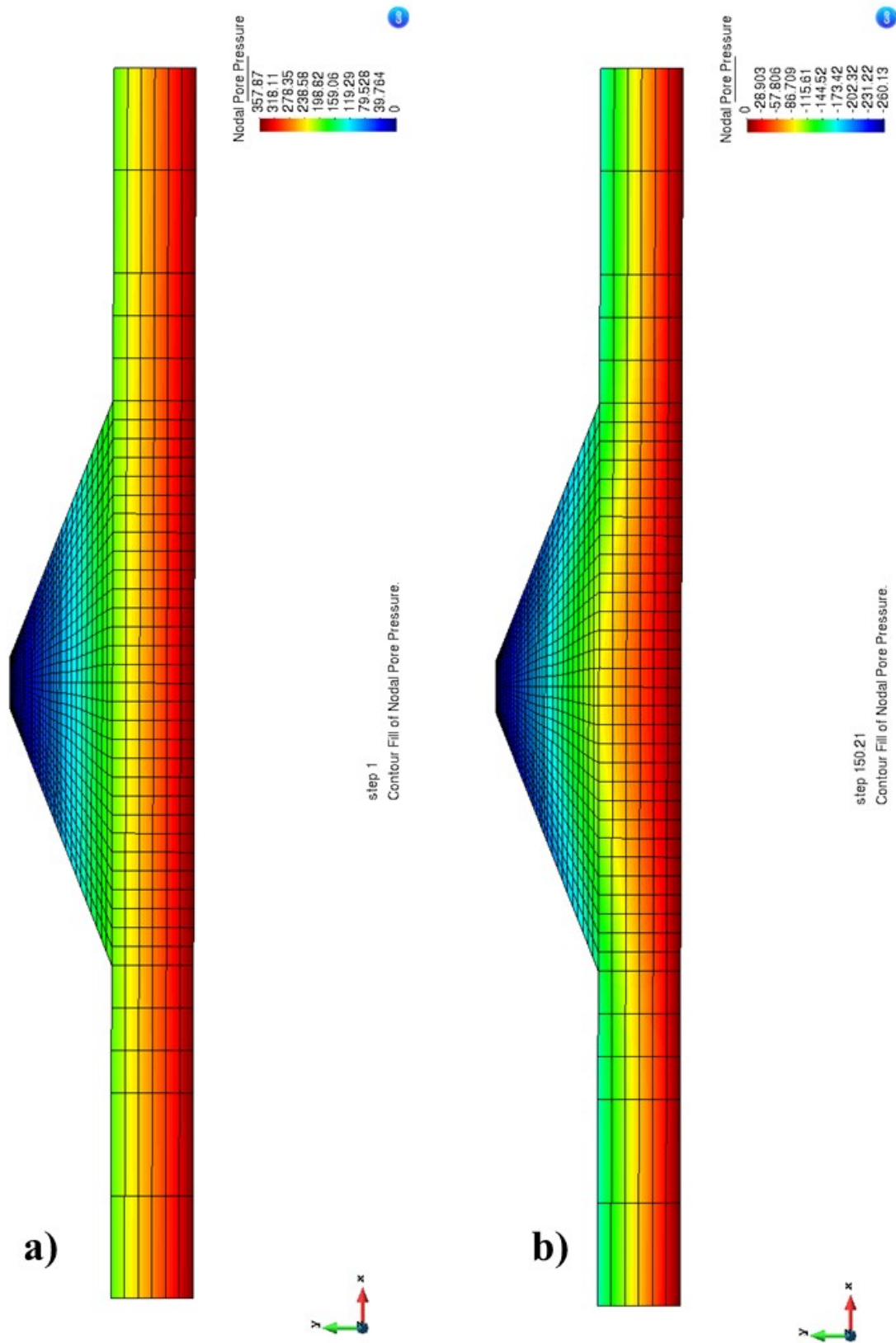


Fig. 5.74 Dam Pore Water Pressure: (a) Hydrostatic Condition and (b) Last Step - New Element

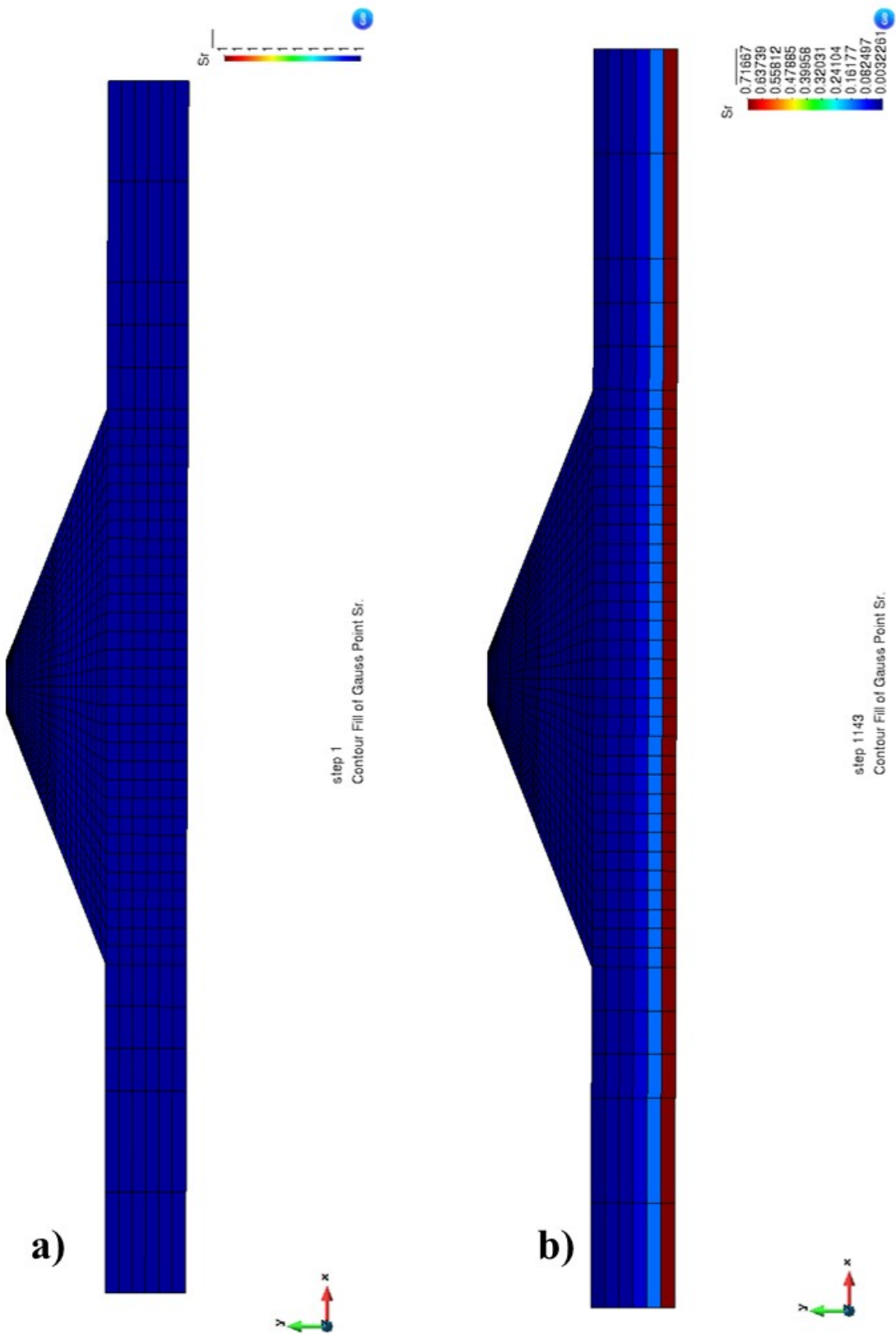


Fig. 5.75 Degree of Saturation: (a) First Step and (b) Last Step - New Element

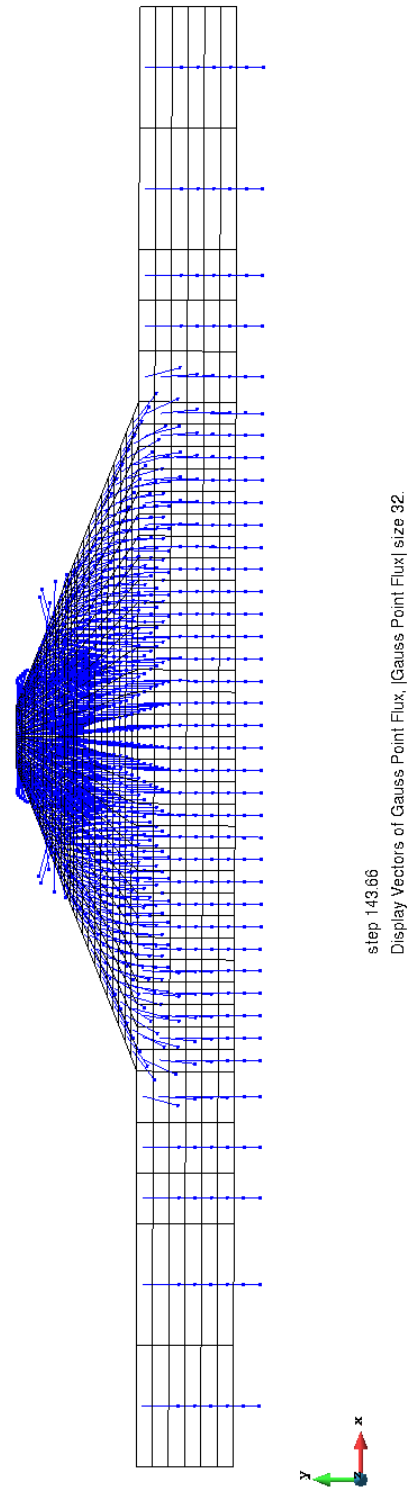


Fig. 5.76 Seepage flux at final stage - New Element

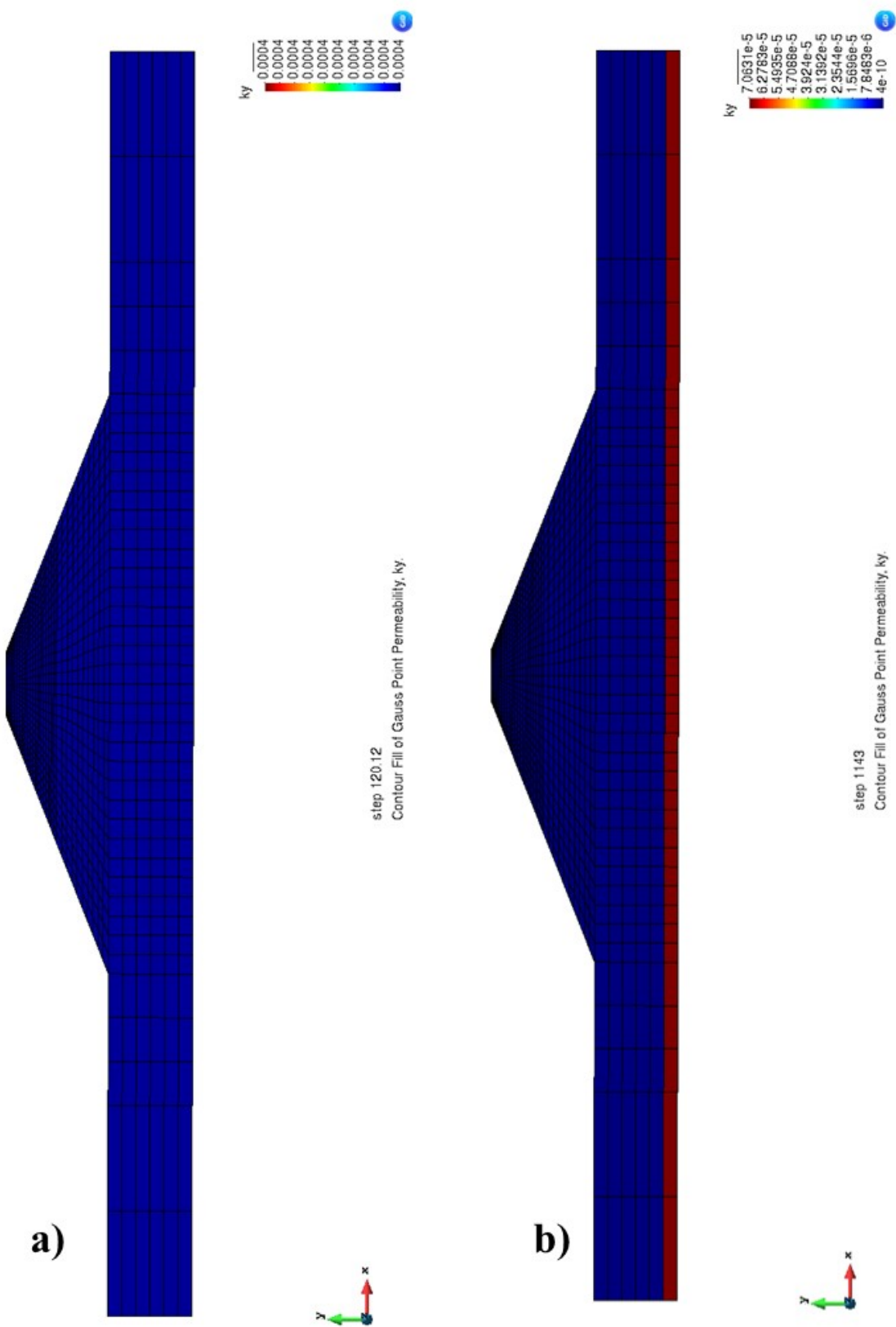


Fig. 5.77 Vertical permeability k_y : (a) First Step and (b) Last Step - New Element



In the transient emptying stage, Figure 5.73(b), the top hydraulic boundary is released and treated as a natural no-flow boundary, whereas the entire base is prescribed as a drained boundary with $p_w = 0$. Therefore, the subsequent seepage process is controlled exclusively by basal drainage.

This configuration represents an idealized but stringent test case in which seepage is driven exclusively by basal drainage under initially saturated conditions. As such, it provides a complementary verification of the robustness of the proposed formulation, particularly with respect to the enforcement of hydraulic boundary conditions, pore-water pressure dissipation, and the coupled hydro-mechanical response under a globally drained base.

Figures 5.74–5.78 summarize the hydraulic response obtained from the seepage analysis of the earth dam with full-base drainage, comparing the initial reference configuration with the final steady-state condition and allowing a clear assessment of the staged hydro-mechanical response.

Figure 5.74 shows the pore-water pressure distribution for (a) the initial hydrostatic state and (b) the final stage after activation of the basal drain. The initial condition exhibits an approximately hydrostatic variation of pore-water pressure with depth, confirming proper hydraulic initialization. After drainage, pore-water pressures decrease progressively toward the base, generating hydraulic gradients directed toward the drained basal boundary.

The degree of saturation contours in Figures 5.75(a) and 5.75(b) indicate an initially fully saturated domain, followed by the development of a partially saturated zone near the basal drainage boundary in the final stage. The upper part of the embankment remains closer to full saturation, reflecting a physically consistent desaturation process driven by pressure relief at the base.

Figure 5.76 presents the seepage flux vectors at the final stage, showing flow convergence toward the downstream base drain and higher flux intensities near the drain. The absence of flow across impermeable boundaries confirms the correct implementation of no-flow conditions.

Figures 5.77 and 5.78 show the evolution of vertical and horizontal hydraulic conductivities, respectively. Both components remain uniform under fully saturated conditions and decrease in the downstream region as partial saturation develops. The consistent trends in k_v and k_h confirm the coherent coupling between saturation and hydraulic properties.

Overall, the results demonstrate that the proposed formulation reliably captures the essential features of seepage through an earth dam with downstream drainage, including the establishment of the initial hydrostatic condition, the stable transition between stages, and the coupled evolution of pore-water pressure, saturation, seepage fluxes, and permeability.

Chapter 6

Verification under Dynamic Loading Conditions

6.1 Introduction

Building upon the static and transient verification presented in Chapter 5, this chapter investigates the performance of the QuadUPSVG element under dynamic loading conditions. The objective is to assess the robustness of the coupled hydro–mechanical formulation in the presence of inertial effects and cyclic loading.

In contrast to the previous chapter, where diffusion-dominated processes were examined, the present verification focuses on wave propagation, dynamic equilibrium, and the interaction between transient pore–pressure response and soil skeleton deformation. Particular attention is given to the consistency of mass coupling terms, the implementation of dynamic boundary conditions, and the stability of the numerical time integration algorithm.

The verification is carried out through the analysis of a soil column subjected to prescribed base excitation. In order to evaluate the robustness of the formulation under inertial effects, the assessment is conducted using an advanced elasto–plastic constitutive description.

The mechanical behaviour is represented exclusively by the Manzari and Dafalias model [30], which provides a bounding-surface plasticity framework suitable for cyclic loading conditions. This formulation allows the interaction between plasticity mechanisms and the coupled pore–pressure response to be assessed consistently under dynamic excitation.

The results obtained with the proposed element are systematically compared with those computed using the reference QuadUP element. The comparison is performed in terms of accelerations, pore–water pressures, stresses, strains, and stress–strain hysteretic response. Through this analysis, the present chapter verifies that the proposed formulation accurately

reproduces the coupled dynamic response of saturated soils and remains stable and consistent under nonlinear cyclic conditions.

6.2 Dynamic response of a saturated soil column under base excitation (ElastoPlastic)

This example investigates the dynamic response of a saturated soil column subjected to prescribed base excitation, with the objective of verifying the robustness of the coupled hydro–mechanical formulation, the time integration strategy, and the implementation of dynamic boundary conditions under *elastoplastic* soil behaviour. The analysis reproduces the free–field one–dimensional soil column configuration adopted in the VELACS centrifuge research program [7], which has become a benchmark reference for validating numerical formulations in earthquake geotechnical engineering.

The problem is formulated in a two–dimensional plane–strain framework using a u – p_w formulation ($\text{ndm}=2$, $\text{ndf}=3$), where the primary unknowns are the solid displacements (u_x, u_y) and the pore–water pressure p_w . In OpenSees, the pore pressure is stored as the third degree of freedom and is retrieved through the `vel` response type.

Figure 6.1 summarizes the initial hydraulic configurations adopted for the dynamic verification. In the fully saturated case, the water table is located above the soil column and the degree of saturation remains approximately equal to unity along the entire depth. In the partially saturated case, the water table is lowered to mid-height of the column. The suction generated in the upper half is converted into degree of saturation through the Van Genuchten retention function adopted in the analysis. Thus, the plotted S_r profile represents the actual initial saturation state used before applying the seismic excitation.

Geometry and discretization. The numerical model represents a vertical soil column of total height 10 m (see Fig. 6.1(a)), extending from the ground surface at $z = 0$ m to a rigid bedrock at $z = -10$ m. The domain is discretized with twenty four quadrilateral elements along the vertical direction, resulting in a uniform element height $\Delta z = 0.5$ m. A unit out–of–plane thickness is assumed. This unit–width idealization is commonly adopted to reproduce one–dimensional site response within a two–dimensional finite element setting.

Kinematic and hydraulic boundary conditions. At the base of the column, the vertical displacement is constrained ($u_y = 0$), while the horizontal motion is driven by the imposed base excitation. Along the lateral boundaries, nodes at the same elevation are tied using `equalDOF` constraints in the displacement degrees of freedom (u_x, u_y) . This constraint

strategy enforces a one-dimensional response by suppressing spurious lateral deformation modes and ensuring identical horizontal and vertical motion across the column width.

Hydraulic conditions reproduce a fully saturated profile. The water table is located 1 m above the ground surface, yielding a hydrostatic pore-pressure distribution throughout the domain. Hydrostatic conditions are imposed through a constant support-motion pattern applied to the pore-pressure degree of freedom at the surface nodes, together with an equivalent vertical surcharge representing the overlying water column.

Staged solution strategy. A staged procedure is adopted. First, a gravity analysis is performed to establish the initial stress state and the hydrostatic pore-pressure distribution. During this initialization phase, the elastic formulation of the constitutive model is adopted through the `updateMaterialStage` command, improving robustness and ensuring a stable equilibrium state. Once gravity convergence is achieved, the material state is updated to activate the elastoplastic response, and the analysis proceeds directly to the dynamic stage while preserving the stress and pore-pressure fields computed during initialization.

Constitutive model. The Dafalias–Manzari constitutive model [30], as implemented in OpenSees [60], is adopted to represent the nonlinear cyclic response of saturated sand. The model is a critical-state-based, bounding-surface plasticity formulation capable of capturing stiffness degradation, irreversible strains, and excess pore-water pressure generation under undrained cyclic loading. The adopted parameters (Table 6.1) represent a medium-dense sand and are consistent with values commonly reported for Nevada sand-type materials reported in Taiebat et al. [137].

Dynamic excitation and reference configuration. The imposed excitation corresponds to the VELACS free-field centrifuge benchmark on saturated sand deposits. A horizontal base acceleration time history is applied (see Figure 6.2), with peak ground acceleration of approximately 0.235 g, total duration of about 19.3 s, and predominant frequency close to 2 Hz. The present model follows the VELACS reference idealization: a vertical column discretized into twenty 0.5 m-thick coupled displacement-pore pressure elements, lateral constraints enforcing equal displacements at the same elevation, and base excitation applied in the horizontal direction. This configuration is routinely used to assess the ability of coupled formulations and constitutive models to reproduce cyclic degradation and liquefaction triggering.

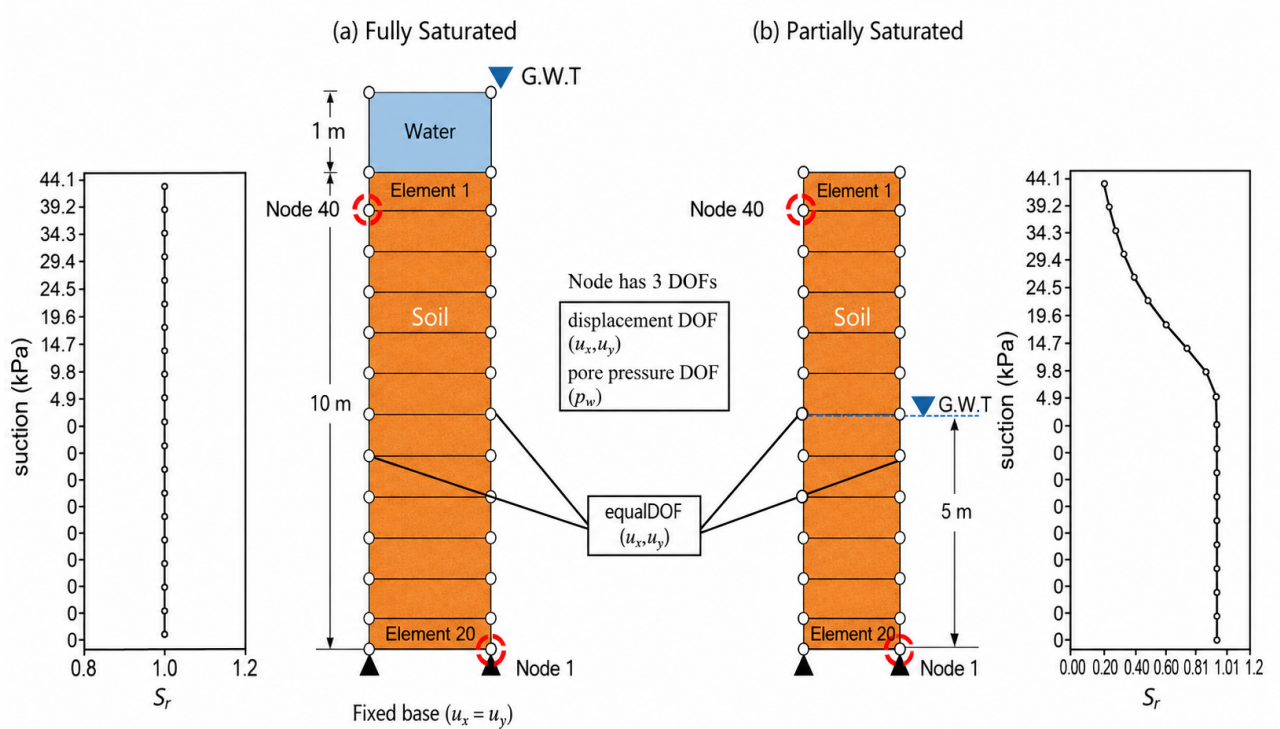


Fig. 6.1 Numerical model representation of the free-field soil column used for dynamic verification: (a) fully saturated configuration and (b) partially saturated configuration.

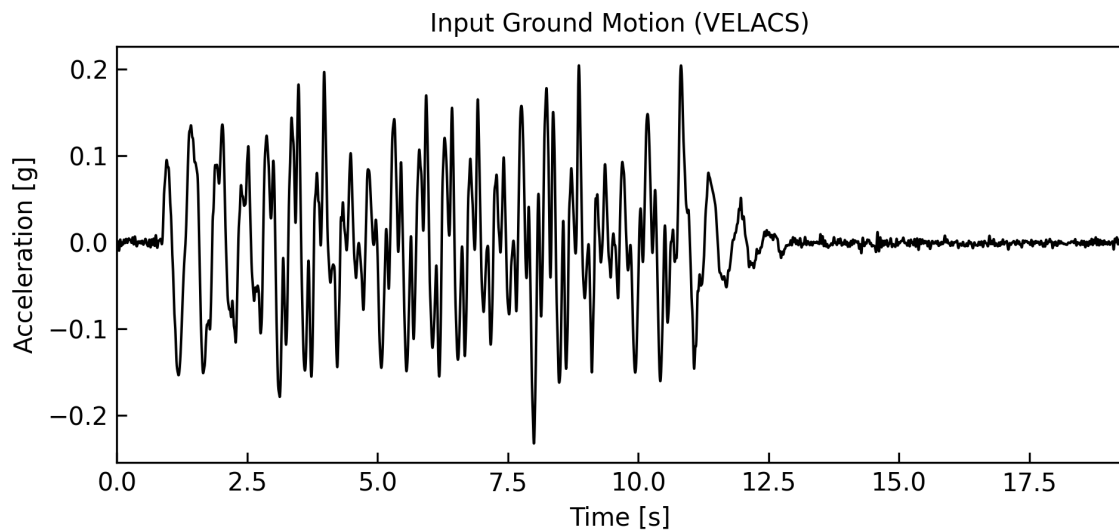


Fig. 6.2 Input Motion

Dynamic analysis settings and output quantities. The dynamic analysis is performed using the Newmark time integration scheme with parameters $\gamma = 0.5$ and $\beta = 0.25$. A constant time step $\Delta t = 5 \times 10^{-3}$ s is adopted, and the analysis is carried out for the duration of the input motion plus an additional 5 s to capture free vibration. Rayleigh damping is introduced to target 3% damping at control frequencies of 2 Hz and 20 Hz.

During shaking, nodal displacements, accelerations, and pore–water pressures are recorded at multiple depths (ground surface and depths of 1, 2.5, 5, 7.5, and 10 m). Element stress and strain responses are also monitored to assess the evolution of effective stresses driven by transient pore–pressure variations.

The dynamic response is evaluated by comparing the reference QuadUP element and the proposed New Element, here treated under fully saturated conditions to verify that it reproduces the same coupled response as QuadUP in the elastoplastic regime. Comparisons are reported at representative nodes and elements located at different depths, and are discussed in terms of kinematic, hydraulic, and mechanical response quantities.

Table 6.1 Dafalias and Manzari [30] constitutive model parameters adopted for the elastoplastic dynamic analysis.

Parameter	Value	Description
G_0	563.04	Elastic shear modulus constant
ν	0.31	Poisson's ratio
M_c	1.25	Critical-state stress ratio (triaxial compression)
c	0.72	Ratio of critical-state stress ratios (extension/compression)
λ_c	0.027	Critical-state line slope
e_0	0.83	Void ratio at reference pressure
ξ	0.45	State parameter constant
m	0.0625	Yield surface parameter
h_0	9.7	Plastic modulus constant
c_h	1.02	Hardening parameter
n_b	1.1	Bounding surface parameter
A_0	0.81	Dilatancy constant
n_d	0.25	Dilatancy exponent
$z_{\max}$	5.0	Fabric-dilatancy parameter
c_z	800	Fabric evolution constant

Figure 6.3 presents the horizontal absolute acceleration time histories recorded at selected depths along the soil column, expressed in units of g . The response exhibits clear nonlinear

features, including amplitude modulation and localized acceleration spikes, particularly during the strong-motion window between approximately 2 and 12 s.

Peak horizontal accelerations show a marked depth dependence. Near the ground surface (Node 1), the response remains limited to about $\pm 0.2 g$. At intermediate depths (Nodes 11 and 22), peaks increase to the order of $\pm 0.6\text{--}0.9 g$, with isolated negative spikes approaching $\sim -1.2 g$. In the lower portion of the column (Nodes 31, 38, and 40), the response becomes more impulsive, with positive peaks close to $\sim 1.0 g$ and intermittent negative spikes reaching approximately $-2.0 g$. After $\sim 12\text{--}13$ s, the acceleration amplitudes decay rapidly at all depths, indicating the end of the strong-motion phase.

The responses obtained with QuadUP and the New Element are virtually coincident at all monitored locations, with no observable phase shift and only negligible amplitude differences, confirming that both formulations consistently reproduce the nonlinear inertial response under fully saturated elastoplastic conditions.

The evolution of pore-water pressure is illustrated in Figure 6.4, reported in kPa. A clear accumulation of excess pore pressure is observed during cyclic loading. At 10 m depth, pore pressure increases from an initial value of approximately 110 kPa to nearly 200 kPa, corresponding to a substantial reduction in effective mean stress. At 7.5 m depth, peak pore pressures reach approximately 160 kPa, while at 5 m depth they reach approximately 110 kPa. Near the surface, peak values remain below 30 kPa.

The maximum excess pore water pressure ratio at 10 m depth, calculated as the ratio between excess pore water pressure Δu and initial vertical effective stress σ'_{vo} , approaches values close to unity, indicating liquefaction conditions during the strongest shaking phase. After approximately 12–15 s, pore pressures stabilize and gradually dissipate. Both elements predict nearly identical pore-pressure histories, confirming consistent hydro–mechanical coupling and effective stress transfer.

Figure 6.6 presents the shear stress–shear strain (τ – γ) hysteretic loops at selected depths. The loops exhibit pronounced nonlinearity, progressive widening, and clear stiffness degradation with increasing cyclic demand.

At 10 m depth, shear stresses reach approximately $\tau_{\max} \approx 55$ kPa and $\tau_{\min} \approx 50$ kPa, while the corresponding shear strains extend up to $\gamma_{\max} \approx 0.5\%$ and $\gamma_{\min} \approx 0.2\%$. The resulting secant shear modulus at peak response is therefore approximately $G_{\text{sec}} \approx \tau_{\max} / \gamma_{\max} \approx 11$ MPa.

At 1 m depth, peak shear stresses reduce to approximately $\pm 20\text{--}22$ kPa, with shear strains reaching approximately $\gamma_{\max} \approx 0.6\%$. Near the surface, stress amplitudes remain below ± 10 kPa due to reduced confinement. The significant enclosed areas of the loops confirm substantial hysteretic energy dissipation through plastic deformation. The loops obtained

with QuadUP and the New Element are practically indistinguishable, with differences in peak stress and strain values remaining negligible.

The stress paths in p – q space are shown in Figure 6.7, with stresses expressed in kPa. A characteristic cyclic “butterfly” pattern develops, particularly pronounced at 10 m depth. During strong shaking, the mean effective stress p reduces significantly while deviatoric stress q oscillates between approximately ± 50 kPa. The contraction of p reflects effective stress loss induced by pore-pressure generation. Shallower layers exhibit similar but smaller excursions in p – q space, consistent with their lower confinement. The near-perfect overlap of the stress paths confirms that both formulations capture the same effective stress evolution under cyclic loading.

Figure 6.8 shows the evolution of vertical strain. Irreversible volumetric deformation clearly develops during shaking.

At 10 m depth, the vertical strain reaches a minimum value of $\epsilon_v^{\min} \approx -2.3 \times 10^{-1}\%$ at the end of the strong motion phase, indicating significant contractive behaviour. No meaningful recovery occurs after shaking, confirming permanent settlement.

At 1 m depth, vertical strain accumulates to approximately $\epsilon_v^{\max} \approx 3.2 \times 10^{-2}\%$, while at the surface the final strain remains on the order of 1.8×10^{-4} . The monotonic trend after approximately 12 s confirms plastic volumetric deformation rather than purely oscillatory behaviour.

Overall, the comparison demonstrates that the proposed New Element, when treated under fully saturated conditions, reproduces the same nonlinear dynamic response as the reference QuadUP formulation. This includes inertial amplification (up to approximately 1.4 g at depth), pore-pressure buildup (up to nearly 200 kPa), effective stress reduction, hysteretic energy dissipation, and permanent deformation. Across all monitored response quantities, differences between the two formulations remain negligible, confirming the robustness and consistency of the proposed element in elastoplastic dynamic analyses.

The evolution of the excess pore pressure ratio r_u is shown in Figure 6.9. A clear depth-dependent response is observed, consistent with the reduction of mean effective stress in the p – q space.

At 10 m depth, r_u increases rapidly during the strong shaking phase (3–10 s), reaching peak values close to unity ($r_u^{\max} \approx 0.98$ –1.00), indicating near-complete loss of initial vertical effective stress. At 7.5 m and 5 m depths, peak values reach approximately 0.95 and 0.90–0.95, respectively. At 2.5 m and 1 m depths, r_u also approaches unity due to the lower initial confinement, although the absolute pore-pressure increments are smaller.

After approximately 12 s, r_u stabilizes and slightly decreases, reflecting partial pore-pressure dissipation once cyclic demand diminishes. The responses obtained with QuadUP

and the proposed New Element are practically the same, with peak differences below 2%, confirming consistent hydro-mechanical coupling and effective stress degradation under fully saturated elastoplastic conditions.

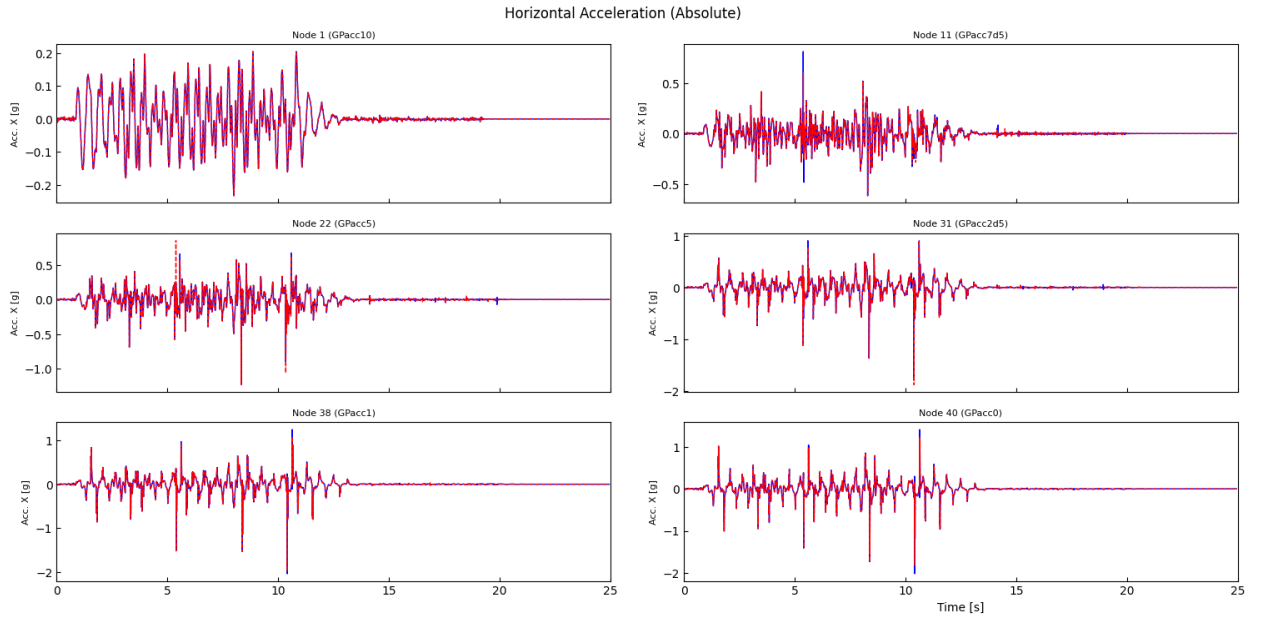


Fig. 6.3 Horizontal acceleration time histories (elastoplastic case).

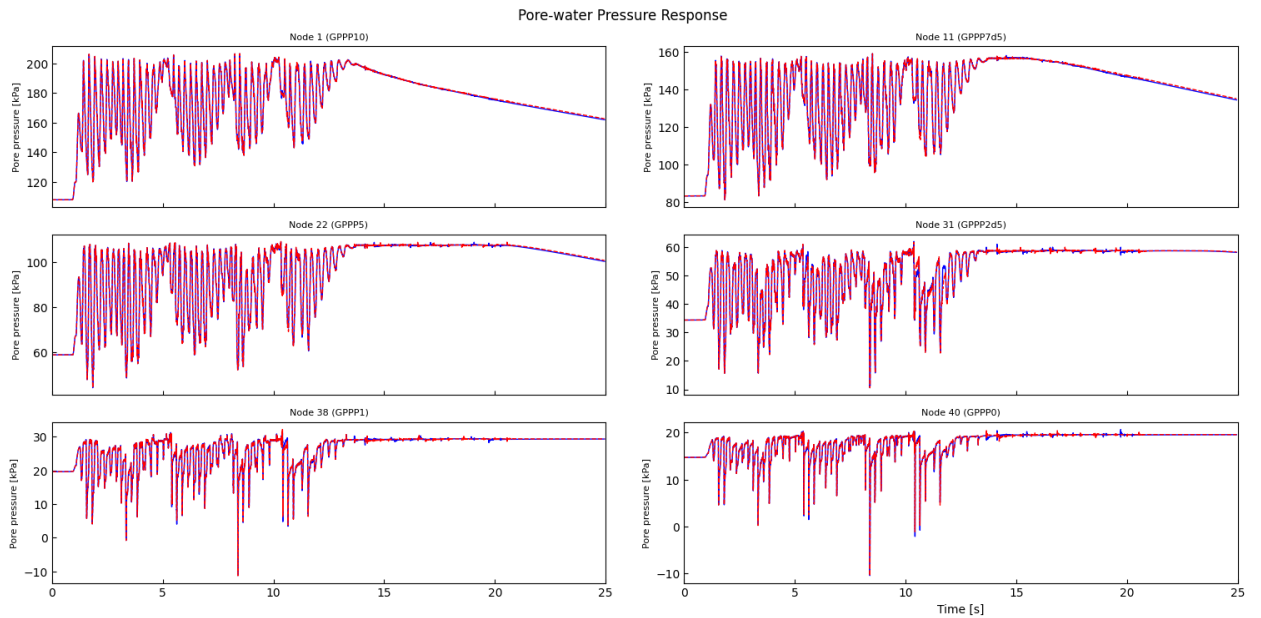


Fig. 6.4 Pore water pressures time histories (elastoplastic case).

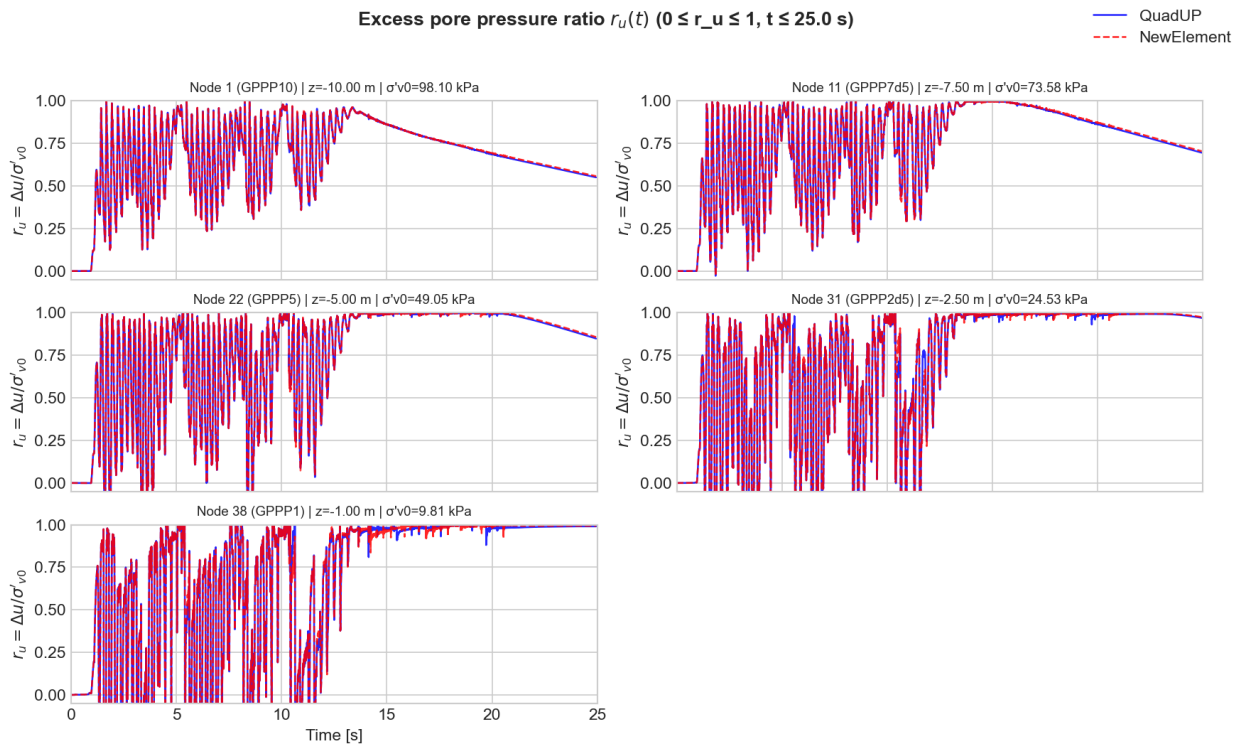
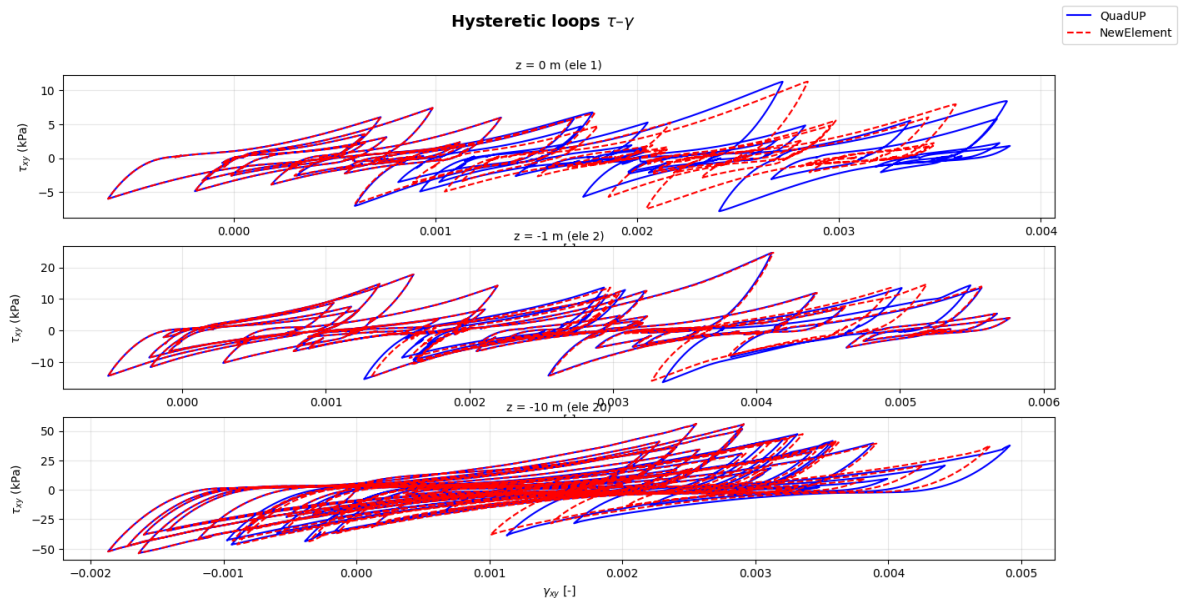
Fig. 6.5 Excess pore pressure ratio r_u .

Fig. 6.6 Hysteretic loops (elastoplastic case).

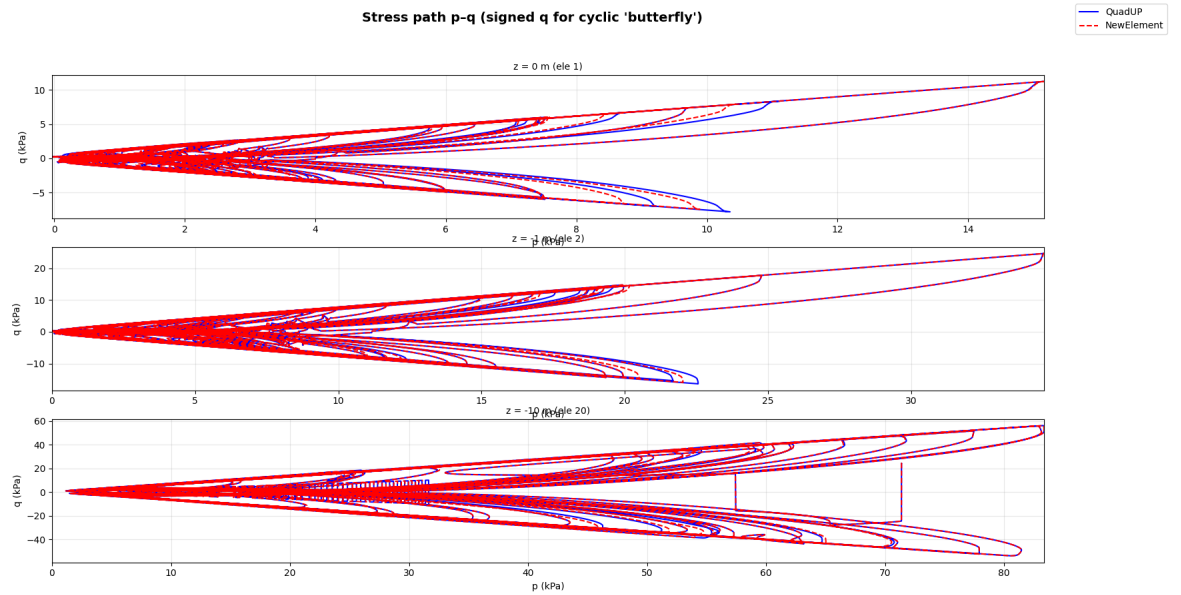


Fig. 6.7 p vs q (elastoplastic case).

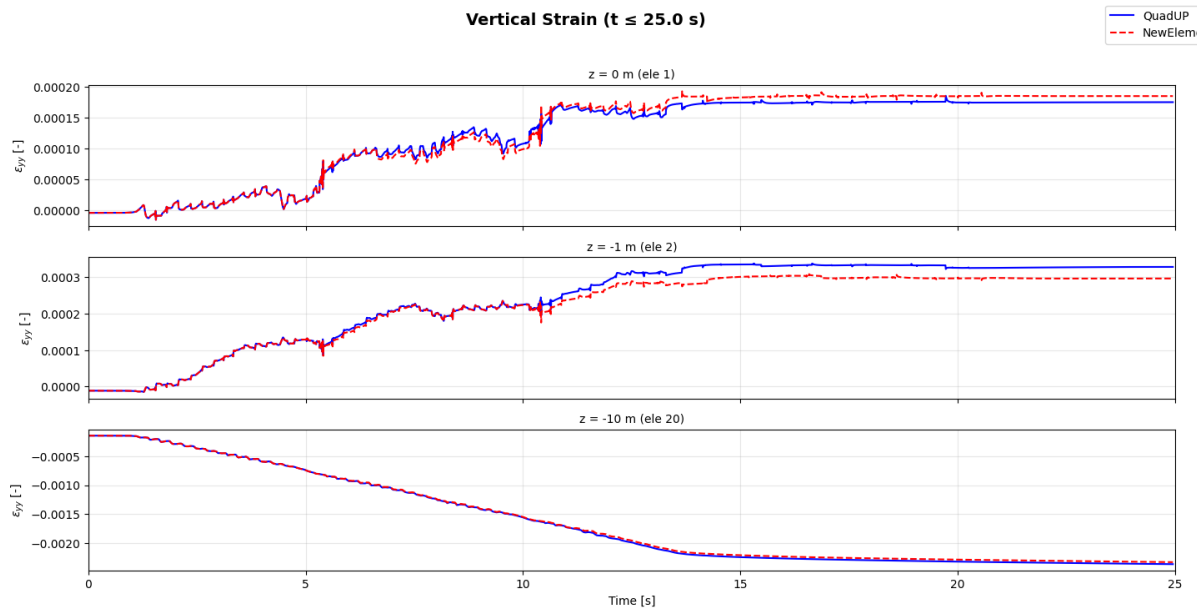


Fig. 6.8 Vertical Strains (elastoplastic case).

6.3 Dynamic response of a partially saturated soil column under base excitation (ElastoPlastic)

This example follows exactly the same numerical framework, geometry, discretization strategy, constitutive model, and dynamic excitation adopted in the previous saturated VELACS benchmark. The sole modification introduced here consists in lowering the phreatic level to mid-height of the soil column prior to the application of the seismic excitation (see Fig. 6.1(b)). All other modelling assumptions and numerical procedures are preserved.

The objective of this configuration is to evaluate the robustness of the coupled hydro-mechanical formulation under partially saturated initial conditions, while maintaining full consistency with the reference VELACS free-field configuration [7]. In particular, this example allows assessment of how the modified initial pore-pressure distribution influences the subsequent elastoplastic dynamic response.

Geometry and discretization. The domain geometry remains unchanged. The model represents a vertical soil column of total height 10 m discretized into twenty quadrilateral coupled u - p_w elements with uniform height $\Delta z = 0.5$ m and unit out-of-plane thickness. The formulation remains two-dimensional plane strain ($ndm=2$, $ndf=3$), with displacement components (u_x, u_y) and pore-water pressure p_w as primary unknowns.

Modified hydraulic configuration. In contrast to the previous fully saturated profile, the water table is now positioned at mid-depth of the column ($z = 5$ m). Consequently:

- The lower half of the column remains fully saturated and exhibits hydrostatic pore-pressure conditions.
- The upper half is initially above the phreatic surface and therefore experiences reduced pore-water pressure.

The hydrostatic distribution is recomputed consistently with the new phreatic level during the gravity initialization stage. No changes are introduced in the kinematic boundary conditions or in the lateral equalDOF constraints enforcing one-dimensional response. For the partially saturated configuration, the soil-water retention behavior follows the Van Genuchten formulation, using the hydraulic parameters $\alpha = 0.0816 \text{ kPa}^{-1}$, $m = 0.083$, and $S_r^{\text{res}} = 0.179$. These parameters define the relationship between the suction generated above the phreatic level and the corresponding degree of saturation. Therefore, the saturation profile shown in Fig. 6.1(b) was not prescribed independently, but was computed from the suction

field generated during the hydraulic staging procedure. For the suction range developed in the upper half of the column, approximately between 0 and 50 kPa, the adopted retention curve produces a progressive reduction of S_r above the phreatic surface, while the lower half remains essentially fully saturated.

It is important to note that the partially saturated profile is not imposed a priori. The analysis initially starts from a fully saturated condition, consistent with the reference configuration. The phreatic level is then progressively lowered to mid-height through hydraulic staging, allowing the pore-pressure field to redistribute naturally under transient flow conditions (as reported in section 5.4). As a consequence, suction develops automatically in the upper half of the column, leading to a mechanically and hydraulically consistent partially saturated state prior to the application of the seismic excitation. This procedure avoids prescribing artificial initial suction distributions and ensures that the initial stress and hydraulic fields are obtained from equilibrium considerations.

Staged solution strategy. The same staged procedure adopted in the saturated benchmark is repeated:

1. Gravity analysis to establish the initial effective stress field and the modified hydrostatic pore-pressure distribution associated with the reduced water table.
2. Activation of the elastoplastic material stage using `updateMaterialStage`.
3. Direct transition to the dynamic phase while preserving the converged stress and pore-pressure state.

Thus, the only difference with respect to the saturated example lies in the initial hydraulic condition.

Constitutive model. The Dafalias–Manzari elastoplastic model [30, 60] is retained without modification. The same parameter set reported in Table 6.1 is adopted, ensuring that differences in response arise exclusively from the altered initial pore-pressure field rather than from changes in constitutive behaviour.

Dynamic excitation. After equilibrium under the modified phreatic condition is achieved, the identical VELACS base acceleration record is applied. The horizontal input motion (PGA ≈ 0.235 g, duration 19.3 s, predominant frequency ≈ 2 Hz) is imposed at the base using the same MultiSupport excitation strategy adopted previously.

Numerical integration and damping. The Newmark time integration scheme ($\gamma = 0.5$, $\beta = 0.25$), constant time step $\Delta t = 5 \times 10^{-3}$ s, and Rayleigh damping calibrated to 3% at 2 Hz and 20 Hz are preserved. Output quantities include nodal displacements, accelerations, pore-water pressures, and element stress–strain histories at the same monitoring depths considered in the reference saturated case.

By construction, this example isolates the influence of the reduced phreatic level on cyclic pore-pressure generation, stiffness degradation, and wave propagation characteristics, while keeping all numerical and constitutive ingredients identical to the fully saturated VELACS configuration.

Figure 6.9 presents the horizontal absolute acceleration time histories recorded at selected depths. The overall dynamic response remains strongly nonlinear, with amplitude modulation and impulsive spikes concentrated within the strong-motion window (approximately 2–12 s), consistent with the saturated benchmark.

A clear depth-dependent amplification pattern is observed. Near the ground surface, peak accelerations remain below approximately $\pm 0.2g$, while intermediate and deeper layers exhibit larger amplitudes approaching $\pm 1.0g$, with occasional negative spikes of greater magnitude. After approximately 12–13 s, acceleration amplitudes decay rapidly, indicating the end of significant inertial excitation.

Despite the modified hydraulic configuration, the acceleration histories predicted by QuadUP and the New Element remain nearly coincident. No observable phase shift is detected, and amplitude differences are negligible. This confirms that the introduction of partially saturated initial conditions does not significantly alter the global inertial response of the column for the excitation level considered.

Figure 6.9 presents the horizontal absolute acceleration time histories recorded at selected depths along the column. The response exhibits clear nonlinear characteristics, including amplitude modulation and localized acceleration spikes concentrated within the strong-motion window between approximately 2 and 12 s. A marked depth dependence is observed. Near the ground surface, peak accelerations remain limited to approximately $\pm 0.2g$. At intermediate depths, amplitudes increase significantly, approaching values close to $\pm 1.0g$, with occasional negative spikes of larger magnitude in the lower portion of the column. After approximately 12–13 s, acceleration amplitudes decay rapidly at all depths, indicating the end of the strong inertial phase.

Despite the modified hydraulic configuration, the acceleration histories obtained with QuadUP and the New Element are practically coincident at all monitored locations. No phase shift or systematic amplitude discrepancy is observed. This indicates that, for the Van Genuchten retention curve adopted in this analysis and for the suction range generated in the

upper half of the column, lowering the phreatic level to mid-height does not significantly alter the global wave propagation or inertial amplification under the present excitation level. The dynamic response remains governed primarily by the overall stiffness and mass distribution, which are only moderately influenced by suction effects at this shaking intensity.

Figure 6.10 shows the evolution of pore-water pressure. In contrast to the fully saturated case, the initial pore-pressure distribution reflects the lowered phreatic level. The lower half of the column starts under hydrostatic positive pore pressure, while the upper half exhibits negative pore pressures (suction) generated naturally during hydraulic staging. During cyclic loading, excess pore pressure develops predominantly in the saturated lower portion. At 10 m depth, pore pressure increases substantially, approaching values comparable to those observed in the saturated benchmark. At intermediate depths, the buildup is slightly reduced, reflecting the influence of the modified initial effective stress state. In the unsaturated upper half, the response differs conceptually between the two formulations. The QuadUP element enforces negative pore pressures primarily to satisfy equilibrium equations and hydraulic continuity, without explicitly incorporating unsaturated constitutive coupling in the stress integration. Conversely, the New Element incorporates the unsaturated hydro-mechanical response through the Van Genuchten retention model and degree-of-saturation dependent coupling. This is particularly relevant because the suction-controlled response is highly nonlinear, since small changes in pore-water pressure may induce significant variations in degree of saturation, hydraulic conductivity, and hydro-mechanical coupling. This nonlinear suction-dependent coupling also affects the mechanical response, as reflected later in the stress paths and strain evolution shown in Figures 6.11, 6.12, and 6.13. As a result, while the overall pore-pressure trends remain similar, noticeable differences appear in the upper layers, particularly during transient suction reduction and partial re-saturation phases. These differences remain moderate and physically consistent with the enhanced formulation.

The shear stress–shear strain hysteretic loops shown in Figure 6.11 exhibit pronounced nonlinear cyclic behaviour and progressive stiffness degradation. At 10 m depth, peak shear stresses reach approximately ± 50 – 55 kPa, with corresponding shear strains on the order of 0.4–0.5%, confirming substantial hysteretic energy dissipation. In the saturated lower portion of the column, both elements produce nearly identical loop shapes. However, in the partially saturated upper layers, subtle differences become visible. The New Element shows modified unloading and reloading branches, reflecting suction-dependent stiffness and effective stress evolution. The QuadUP formulation, while capable of generating negative pore pressures, does not incorporate suction-dependent mechanical hardening in the stress integration algorithm, which explains the minor discrepancies observed in loop geometry.

The stress paths in p – q space (Figure 6.12) further clarify this behaviour. In the saturated lower layers, both formulations exhibit the characteristic cyclic “butterfly” pattern associated with progressive reduction of mean effective stress p and oscillations of deviatoric stress q up to approximately ± 50 kPa. This confirms that, under fully saturated conditions, the New Element reproduces the response of the reference formulation satisfactorily. In the upper half of the column, initial suction modifies the effective stress state. The New Element accounts for this contribution directly in the hydro–mechanical coupling, resulting in slightly different excursions of p during cyclic loading. These differences become more evident in the partially saturated region, where the highly nonlinear relation between suction, degree of saturation, and effective stress starts to influence the mechanical response. Although QuadUP reproduces negative pore pressures at the hydraulic level, it does not fully reflect suction effects within the constitutive stress integration, leading to minor but consistent differences in the p – q trajectories during early cycles when suction reduction is most active.

Figure 6.13 presents the evolution of vertical strain. Permanent volumetric deformation develops primarily in the saturated lower half of the column, consistent with cyclic contractive behaviour and pore-pressure accumulation. At 10 m depth, irreversible compressive strains of the order of 10^{-3} are observed by the end of shaking. In the upper partially saturated region, vertical strains remain smaller in magnitude and show slight differences between formulations. The New Element, through suction-dependent stiffness modulation, predicts marginally different volumetric strain accumulation compared to QuadUP, which does not include explicit unsaturated mechanical coupling. A similar interpretation applies to Figure 6.13, where the differences between the two formulations are mainly associated with the nonlinear suction-dependent response of the partially saturated zone.

Overall, the comparison demonstrates that in the saturated portion of the column both formulations produce virtually identical results across all response quantities. Therefore, the proposed element behaves consistently with the classical saturated formulation when the soil remains saturated. In the partially saturated region, differences remain limited but physically meaningful. These differences appear because, once suction is present, the response is governed by a highly nonlinear hydro–mechanical coupling that affects not only the pore-pressure field but also the mechanical variables shown in Figures 6.11, 6.12, and 6.13. The New Element captures suction-dependent hydro–mechanical coupling, whereas QuadUP reproduces negative pore pressures primarily as a hydraulic variable satisfying equilibrium. Importantly, these differences do not compromise numerical stability or global consistency. The partially saturated configuration therefore confirms the robustness of the proposed element and highlights its capability to extend the classical u – p_w formulation toward physically consistent unsaturated dynamic analyses.

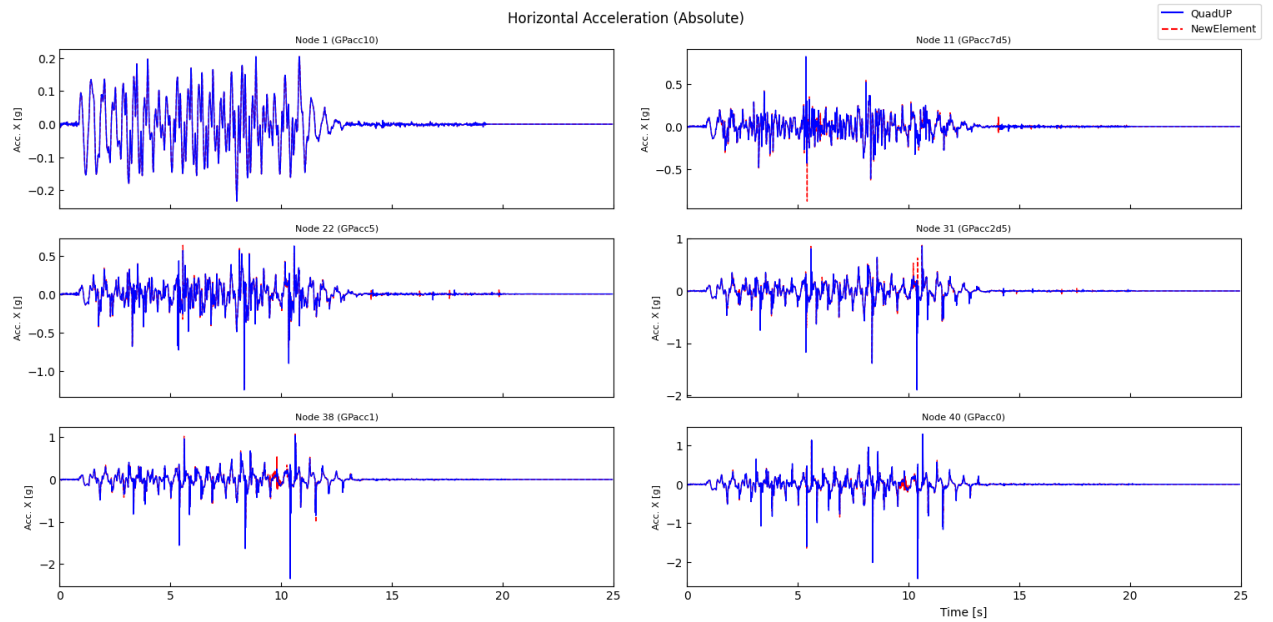


Fig. 6.9 Horizontal acceleration time histories (elastoplastic case).

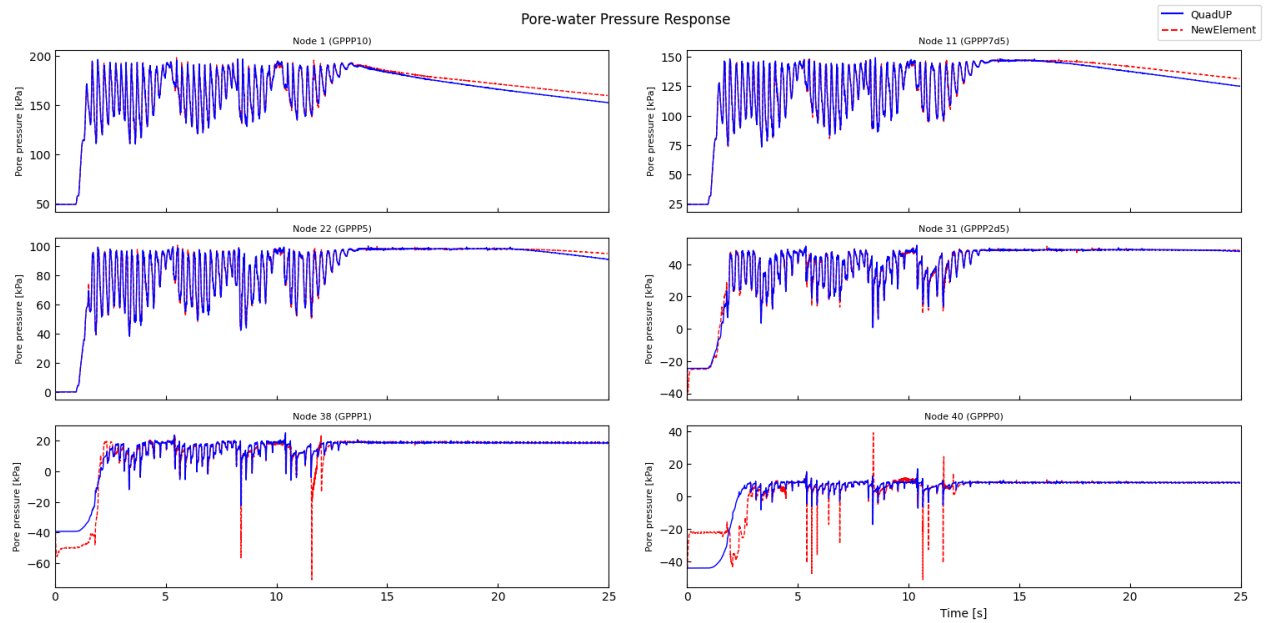


Fig. 6.10 Pore water pressures time histories (elastoplastic case).

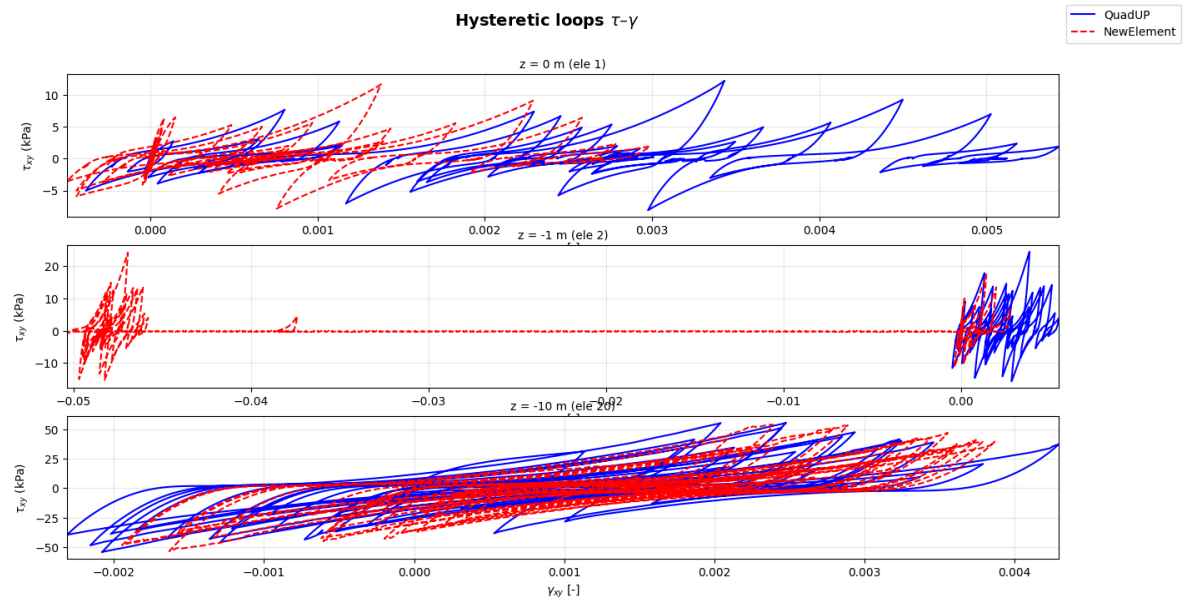
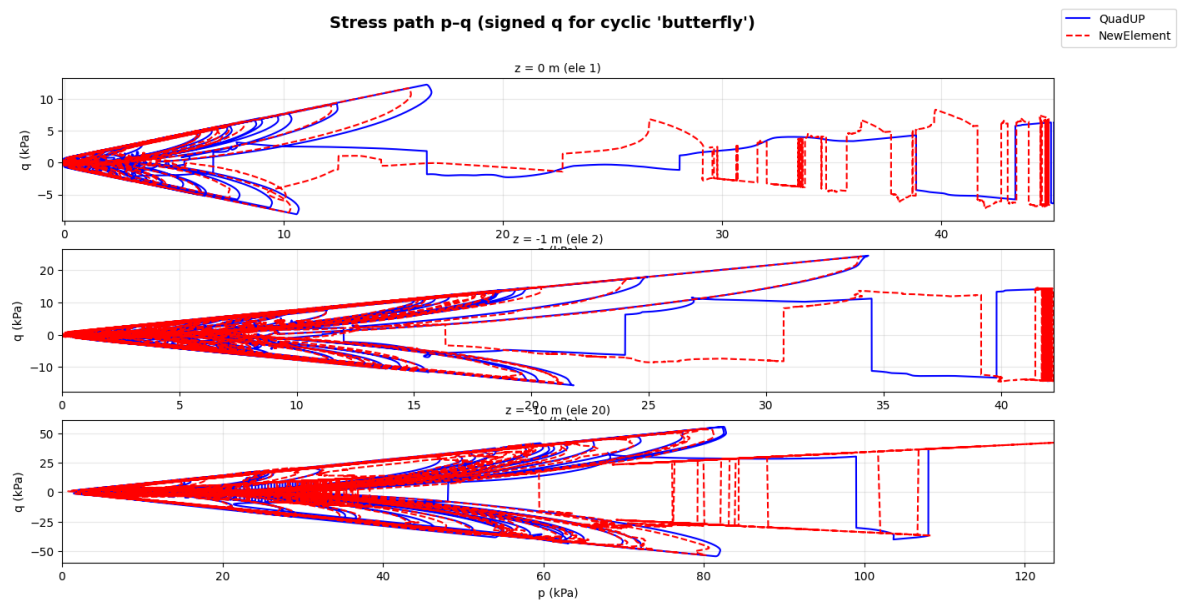


Fig. 6.11 Hysteretical loops (elastoplastic case).

Fig. 6.12 p vs q (elastoplastic case).

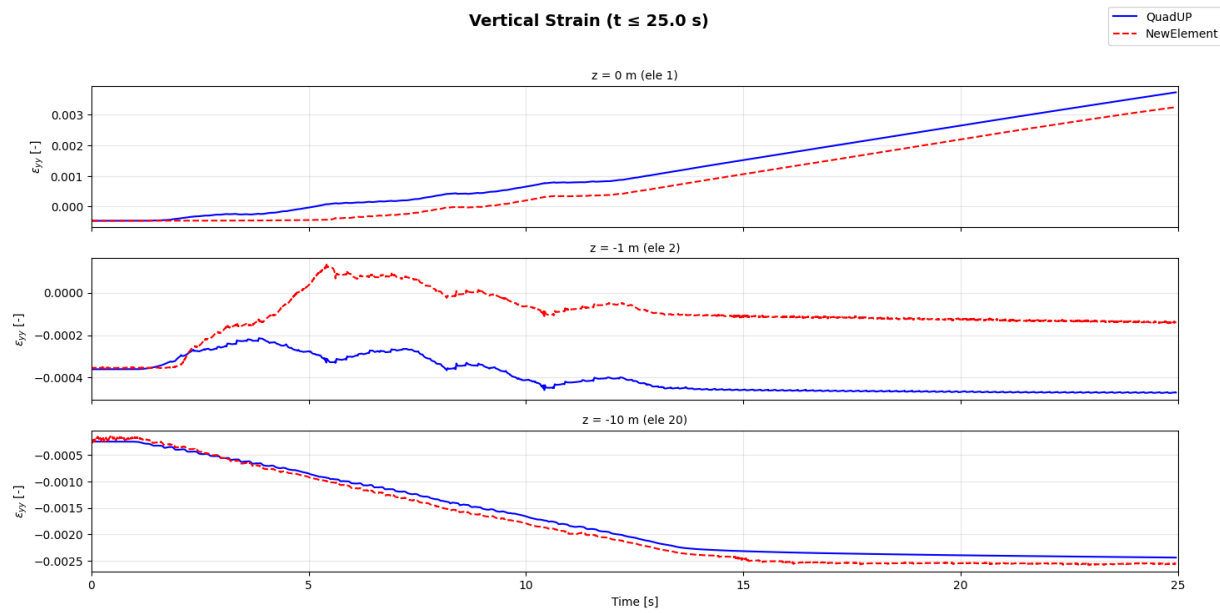


Fig. 6.13 Vertical Strains (elastoplastic case).

Conclusions and Recommendations

Conclusions

The present work addressed the numerical modeling of coupled hydro–mechanical behavior of soils, with particular emphasis on partially saturated conditions and their implementation within the OpenSees framework. Based on the theoretical developments, numerical implementation, and verification analyses presented in Chapters 2–6, the following main conclusions can be drawn:

- **Rigorous extension of the classical u – p_w formulation.** The proposed FourNodeQuadUPSVG element preserves the theoretical structure of the original QuadUP formulation while consistently introducing suction-dependent coupling terms. The modification does not alter the underlying mixed finite element structure, ensuring numerical consistency and compatibility with existing solution strategies.
- **Consistent hydro–mechanical coupling under partial saturation.** The incorporation of the van Genuchten retention relationship enabled a continuous description of degree of saturation, permeability evolution, and effective stress contributions. This allowed a smooth transition between saturated and unsaturated regimes without artificial switching criteria.
- **Independent implementation and verification of the constitutive model.** The ManzariDafaliasIsoHard constitutive model was implemented and verified independently prior to its use in boundary value analyses. This ensured that the observed global responses result from the coupled formulation rather than from numerical artifacts of the constitutive integration.
- **Verification across multiple boundary value problems.** The formulation was validated through: (i) one-dimensional consolidation, (ii) transient partially saturated column analyses, (iii) seepage through earth-dam configurations with drainage boundaries, and (iv) dynamic response of a saturated soil column under base excitation. In all

cases, the results showed consistency with reference solutions and expected physical trends.

- **Elastic response under seepage-dominated loading.** For purely hydraulic loading conditions, stress paths remained within the elastic domain of the adopted constitutive framework. This confirms that plastic mechanisms are not artificially triggered by hydraulic transients.
- **Capability for fully coupled dynamic analysis.** The element accurately captured wave propagation, inertial effects, and pore-pressure evolution within a unified hydro-mechanical framework, demonstrating its suitability for seismic applications.
- **Computational extensibility.** The modular implementation within OpenSees facilitates future integration of more advanced constitutive laws, alternative retention models, and extended multi-physics couplings.

Overall, the developed element-model framework constitutes a consistent and numerically robust platform for advanced coupled hydro-mechanical analyses in saturated and partially saturated soils.

Recommendations and Future Work

Although the objectives of this work were successfully achieved, several extensions may significantly enhance the predictive capability and scope of the proposed formulation:

- **Activation of plastic mechanisms under strong cyclic loading.** Future studies should investigate the fully nonlinear response of partially saturated soils under cyclic and seismic loading, including the assessment of liquefaction resistance and cyclic mobility within the implemented stress-ratio framework.
- **Suction-dependent hardening laws.** Incorporating suction directly into the hardening evolution equations of bounding surface plasticity models would allow a more realistic description of stiffness degradation and cyclic accumulation effects.
- **Enhanced permeability evolution.** Coupling hydraulic conductivity with evolving porosity and plastic volumetric strain may improve predictions in problems involving large deformations or progressive collapse mechanisms.

- **Experimental and centrifuge validation.** Systematic comparison with laboratory and centrifuge test data under controlled partially saturated cyclic conditions would provide additional confidence in the predictive capabilities of the formulation.
- **Application to performance-based earthquake engineering.** The framework may be integrated into hazard assessment studies to evaluate the influence of partial saturation on site response, excess pore-pressure generation, and seismic performance of earth structures.

The present work establishes a solid computational foundation for the simulation of coupled hydro–mechanical processes in unsaturated soils and provides a platform for continued research in advanced computational geomechanics.

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Appendix A

Implementation FourNodeQuadUP element

A.1 getMass() Function

A.1.1 Displacement Mass Matrix $\mathbf{M}$

The consistent displacement mass matrix $\mathbf{M}$ for saturated porous media is defined as:

$$\mathbf{M} = \int_{\Omega} (\mathbf{N}^u)^T \rho \mathbf{N}^u d\Omega \quad (\text{A.1})$$

where:

- $\mathbf{N}^u$: matrix of shape functions associated with nodal displacements (u_x, u_y) ,
- ρ : bulk density of the mixture (solid + fluid),
- Ω : domain of the element.

In OpenSees, the integral is evaluated using 4-point Gauss quadrature. The discrete form becomes:

$$M_{ij} \approx \sum_{m=1}^4 \rho^{(m)} N_i^u(x_m) N_j^u(x_m) |J(x_m)| w^{(m)} = \sum_{m=1}^4 dvol[m] \cdot \rho^{(m)} \cdot N_i^u(x_m) N_j^u(x_m) \quad (\text{A.2})$$

where:

- $dvol[m] = |J(x_m)| w^{(m)}$: Jacobian determinant times Gauss weight at point m ,

- $\rho^{(m)}$ is evaluated by `mixtureRho(m)` in code,
- N_i^u are shape functions for displacements.

In the `getMass()` function:

Listing A.1 Displacement mass matrix **M** assembly in `getMass()`

```
for (int i = 0, i1 = 0; i < 12; i += 3, ++i1) {
  for (int j = 0, j1 = 0; j < 12; j += 3, ++j1) {
    for (int m = 0; m < 4; ++m) {
      double Nrho = dvol[m] * mixtureRho(m)
        * shp[2][i1][m] * shp[2][j1][m];
      K(i, j)      += Nrho; // u_x - u_x
      K(i+1, j+1) += Nrho; // u_y - u_y
    }
  }
}
```

This fills the $\mathbf{M} \in \mathbb{R}^{8 \times 8}$ block in the global matrix corresponding to displacement DOFs at nodes 1–4 (DOFs 0–1, 3–4, 6–7, 9–10).

A.1.2 Pore Pressure Compressibility Matrix **S**

The fluid compressibility matrix is defined as:

$$\mathbf{S} = - \int_{\Omega} \frac{1}{K_c} (\mathbf{N}^p)^T \mathbf{N}^p d\Omega \quad (\text{A.3})$$

where:

- $\mathbf{N}^p$: shape function vector for pore pressure DOFs,
- K_c : fluid bulk modulus.

In OpenSees, this is computed similarly:

$$S_{ij} \approx - \sum_{m=1}^4 \frac{1}{K_c} N_i^p(x_m) N_j^p(x_m) dvol[m] \quad (\text{A.4})$$

Listing A.2 Compressibility matrix **S** assembly in `getMass()`

```
double oneOverKc = 1.0 / kc;
for (int i = 2, i1 = 0; i < 12; i += 3, ++i1) {
```

```

for (int j = 2, j1 = 0; j < 12; j += 3, ++j1) {
    for (int m = 0; m < 4; ++m) {
        K(i, j) += -dvol[m] * oneOverKc
                  * shp[2][i1][m] * shp[2][j1][m];
    }
}
}

```

This loop fills the $\mathbf{S} \in \mathbb{R}^{4 \times 4}$ block, corresponding to pore pressure DOFs at indices 2, 5, 8, and 11.

A.1.3 Global Mass Matrix Structure

The final assembled mass matrix in `getMass()` has the block-diagonal structure:

$$\mathbf{K}_{\text{mass}} = \begin{bmatrix} \mathbf{M} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix} \in \mathbb{R}^{12 \times 12} \quad (\text{A.5})$$

where:

- $\mathbf{M} \in \mathbb{R}^{8 \times 8}$: displacement mass matrix,
- $\mathbf{S} \in \mathbb{R}^{4 \times 4}$: compressibility matrix for pore pressure,
- Off-diagonal blocks are zero in this function; coupling matrices are added elsewhere (e.g., in `getDamp()`).

A.1.4 `getDamp()` Function

A.1.5 Assembly of Mechanical Damping and Mass-Proportional Terms

Listing A.3 Rayleigh damping and mass term assembly in `getDamp()`

```

if (betaK != 0.0)
    Kdamp.addMatrix(1.0, this->getTangentStiff(), betaK);
if (betaK0 != 0.0)
    Kdamp.addMatrix(1.0, this->getInitialStiff(), betaK0);
if (betaKc != 0.0)
    Kdamp.addMatrix(1.0, *Kc, betaKc);

```

```

if (alphaM != 0.0) {
  this->getMass();
  for (i = 0; i < 12; i += 3) {
    for (j = 0; j < 12; j += 3) {
      Kdamp(i,j)      += K(i,j)      * alphaM;
      Kdamp(i+1,j+1)  += K(i+1,j+1)  * alphaM;
    }
  }
}

```

$$\mathbf{C}_{\text{modified}} = \alpha_M \mathbf{M} + \beta_K \mathbf{K}_T + \beta_{K_0} \mathbf{K}_0 + \beta_{K_c} \mathbf{K}_c \quad (\text{A.6})$$

Interpretation:

- $\alpha_M \mathbf{M}$ is the mass-proportional damping matrix;
- $\beta_K, \beta_{K_0}, \beta_{K_c}$ scale the tangent, initial, and committed stiffness matrices;
- This corresponds to the upper-left block of $\mathbf{K}_{\text{damp}}$.

A.1.6 Assembling the Coupling Matrix $\mathbf{Q}$

The coupling matrix $\mathbf{Q}$ links the solid skeleton velocity to the fluid pressure field, and is theoretically defined as:

$$Q_{ij} = - \int_{\Omega} (\nabla N_i^u)^T N_j^p d\Omega = - \int_{\Omega} \left[\frac{\partial N_i^u}{\partial x}, \frac{\partial N_i^u}{\partial y} \right] N_j^p d\Omega \quad (\text{A.7})$$

In OpenSees, this is implemented using numerical Gauss integration as follows:

Listing A.4 Discrete evaluation of $\mathbf{Q}_{ij}$ in getDamp()

```

for (i = 0; i < 12; i += 3) {
  i1 = i / 3;
  for (j = 2; j < 12; j += 3) {
    j1 = (j - 2) / 3;
    for (m = 0; m < 4; m++) {
      Kdamp(i,j)  += -dvol[m]*shp[0][i1][m]*shp[2][j1][m]; // dNu/dx
      * Np
    }
  }
}

```

```

        Kdamp(i+1,j) += -dvol[m]*shp[1][i1][m]*shp[2][j1][m]; // dNu/dy
    * Np
    }
    Kdamp(j,i) = Kdamp(i,j); // symmetry
    Kdamp(j,i+1) = Kdamp(i+1,j);
}
}

```

Interpretation:

- shp[0][i1][m] corresponds to $\frac{\partial N_i^u}{\partial x}$ at Gauss point m ,
- shp[1][i1][m] corresponds to $\frac{\partial N_i^u}{\partial y}$,
- shp[2][j1][m] evaluates N_j^p at Gauss point m ,
- dvol[m] contains the Gauss weight times Jacobian determinant.

Thus, the integral is approximated by:

$$Q_{ij} \approx - \sum_{m=1}^4 dvol[m] \left(\frac{\partial N_i^u}{\partial x}(x_m) N_j^p(x_m) + \frac{\partial N_i^u}{\partial y}(x_m) N_j^p(x_m) \right) \quad (\text{A.8})$$

A.1.7 Assembly of the Permeability Matrix $\mathbf{H}$

The matrix $\mathbf{H}$ arises from Darcy's law and governs the flow resistance as a function of pore pressure gradients:

$$H_{ij} = - \int_{\Omega} \nabla N_i^p \cdot \mathbf{k} \cdot \nabla N_j^p d\Omega \quad (\text{A.9})$$

where $\mathbf{k} = \text{diag}(k_x, k_y)$ is the intrinsic permeability tensor.

In OpenSees, this is implemented as:

Listing A.5 Discrete evaluation of $\mathbf{H}_{ij}$ in getDamp()

```

for (i = 2; i < 12; i += 3) {
    int i1 = (i - 2) / 3;
    for (j = 2; j < 12; j += 3) {
        int j1 = (j - 2) / 3;
        for (m = 0; m < 4; m++) {
            Kdamp(i,j) += - dvol[m] * (
                perm[0]*shp[0][i1][m]*shp[0][j1][m] +

```

```

        perm[1]*shp[1][i1][m]*shp[1][j1][m]
    );
}
}
}

```

Interpretation:

- $\text{shp}[0][i1][m] = \frac{\partial N_i^p}{\partial x}$,
- $\text{shp}[1][i1][m] = \frac{\partial N_i^p}{\partial y}$,
- $\text{perm}[0], \text{perm}[1]$ are k_x and k_y ,
- The expression matches a numerical evaluation of the permeability term.

Hence, the approximation:

$$H_{ij} \approx - \sum_{m=1}^4 d\text{vol}[m] \left(k_x \frac{\partial N_i^p}{\partial x} \frac{\partial N_j^p}{\partial x} + k_y \frac{\partial N_i^p}{\partial y} \frac{\partial N_j^p}{\partial y} \right) \quad (\text{A.10})$$

A.1.8 Structure of the Matrix $\mathbf{K}$

The matrix assembled and returned by `getDamp()` has the block structure:

$$\mathbf{K} = \begin{bmatrix} \mathbf{C}_{\text{modified}} & \mathbf{Q} \\ \mathbf{Q}^T & \mathbf{H} \end{bmatrix} \in \mathbb{R}^{12 \times 12} \quad (\text{A.11})$$

This corresponds to the general time-dependent formulation for u–p elements in saturated porous media mechanics.

A.2 getInitialStiff() Function

A.2.1 Initial Stiffness Matrix

The initial stiffness matrix $\mathbf{K}_0$ for a u–p element is derived from the principle of virtual work and is given by:

$$\mathbf{K}_0 = \int_{\Omega} \mathbf{B}^T \mathbf{D}_0 \mathbf{B} d\Omega \quad (\text{A.12})$$

where:

- $\mathbf{B}$ is the strain-displacement matrix containing derivatives of shape functions,
- $\mathbf{D}_0$ is the initial elastic constitutive matrix (tangent stiffness),
- Ω is the domain of the finite element.

In OpenSees, this matrix is computed in the function `getInitialStiff()` using Gauss quadrature. The expression is approximated as:

$$K_{ia,ib} \approx \sum_{m=1}^{n_{gp}} dvol[m] \cdot \mathbf{B}_\alpha^T \mathbf{D}_0^{(m)} \mathbf{B}_\beta \quad (\text{A.13})$$

The corresponding C++ snippet from `FourNodeQuadUP.cpp` is:

Listing A.6 Assembly of $\mathbf{K}_0$ in `getInitialStiff()`

```
for (int i = 0; i < 4; i++) {
    const Matrix &D = theMaterial[i]->getInitialTangent();

    for (int alpha = 0, ia = 0; alpha < 4; alpha++, ia += 3) {
        for (int beta = 0, ib = 0; beta < 4; beta++, ib += 3) {
            // DB = D * B * dvol
            DB[...] = ...;
            K(ia,ib) += shp[0][alpha][i]*DB[0][0] +
shp[1][alpha][i]*DB[2][0];
            K(ia,ib+1) += shp[0][alpha][i]*DB[0][1] +
shp[1][alpha][i]*DB[2][1];
            K(ia+1,ib) += shp[1][alpha][i]*DB[1][0] +
shp[0][alpha][i]*DB[2][0];
            K(ia+1,ib+1) += shp[1][alpha][i]*DB[1][1] +
shp[0][alpha][i]*DB[2][1];
        }
    }
}
```

Each node has 2 displacement DOFs: u_x and u_y . For a 4-node element:

- Total displacement DOFs: $4 \times 2 = 8$
- Therefore, $\mathbf{K}_0 \in \mathbb{R}^{8 \times 8}$

Same demonstration form tangent stiffness matrix $\mathbf{K}_T$